# CONVERGENCE OF A DISCONTINUOUS GALERKIN METHOD FOR THE MISCIBLE DISPLACEMENT UNDER MINIMAL REGULARITY

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**Abstract.** Discontinuous Galerkin time discretizations are combined with the mixed finite element and continuous finite element methods to solve the miscible displacement problem. Stable schemes of arbitrary order in space and time are obtained. Under minimal regularity assumptions on the data, convergence of the scheme is proved by using compactness results for functions that may be discontinuous in time.

**Key words.** high order time discretization, mixed method, finite element method, stability, compactness

## 1. Introduction.

In the tertiary oil recovery process a polymeric solvent mixes with the trapped oil in the reservoir and the fluid mixture is forced out of the reservoir. The resulting flow problem is characterized by the miscible displacement equations for which the flow is of Darcy type, i.e. the fluid velocity $\mathbf{u}$ is proportional to the gradient of the fluid pressure $\varphi$, and the velocity satisfies the continuity equation in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, over a time interval $(0, T)$:

$$\mathbf{u} = -\mathbf{K}(c)\nabla\varphi, \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

$$\operatorname{div}(\mathbf{u}) = f, \quad \text{in} \quad \Omega \times (0,T). \tag{1.2}$$

The matrix $\mathbf{K}(c)$ depends on the solvent concentration $c$ and the spatial variable $x$. For readability we only write explicitely the dependence on $c$.

$$\mathbf{K}(c) = \frac{1}{\mu(c)}\mathbf{k}(x).$$

The permeability matrix $\mathbf{k}$ measures the resistance of the porous medium to the flow and it may vary rapidly in space. The fluid viscosity $\mu(c)$ is a nonlinear function of the solvent concentration and usually follows a quarter-power mixing law [13].

The balance of mass for the solvent gives the following transport equation for the concentration,

$$\phi c_t - \operatorname{div}\left(D(\mathbf{u})\nabla c - \mathbf{u}c\right) = g, \quad \text{in} \quad \Omega \times (0,T). \tag{1.3}$$

The porosity $\phi$ denotes the fraction of volume available for flow. The matrix $D(\mathbf{u})$ is a diffusion-dispersion tensor given by the semi-empirical relation

$$D(\mathbf{u}) = \phi d_m I + |\mathbf{u}|\left(\alpha_l E(\mathbf{u}) + \alpha_t(I - E(\mathbf{u}))\right), \tag{1.4}$$

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where $E(\mathbf{u}) = \mathbf{u}\mathbf{u}^T/|\mathbf{u}|^2$ and $|\mathbf{u}|$ is the Euclidean norm of $\mathbf{u}$. The coefficients are the molecular diffusion $d_m$, the longitudinal dispersivity $\alpha_l$ and the transverse dispersivity $\alpha_t$ and may depend upon space. This important problem is particularly interesting in the case where the solvent viscosity is higher than the resident fluid viscosity, which leads to fingering phenomena.

The problem (1.1)-(1.3) is completed by an initial concentration $c_0$ defined in $\Omega$ and by boundary conditions prescribed below.

In this work, we analyze a class of numerical schemes for the solution of this system which employ mixed finite elements for the Darcy system, classical Galerkin methodology for the concentration equation, and use discontinuous Galerkin (DG) time discretizations. This class admits schemes of arbitrarily high order in space and time, and we establish convergence when (i) the coefficients $\mathbf{k}$, $\phi$, $d_m$, $\alpha_l$ and $\alpha_t$ are not smooth, (ii) the diffusion-dispersion matrix $D(\mathbf{u})$ is unbounded, and (iii) no maximum principle is available for the numerical solutions. To our knowledge, this paper presents the first high order in time stepping schemes for solving the miscible displacement equation. In addition, convergence is proved via a compactness argument which requires minimal assumptions on the data. Convergence of the numerical schemes establishes existence of (weak) solutions.

Bounds upon the concentration follow from a delicate coupling of the flow and concentration equations which motivates the mixed formulation and time DG stepping considered here. Monotonicity of the elliptic term $-\mathrm{div}(D(\mathbf{u})\nabla c - c\mathbf{u})$ is also essential, and since the diffusion matrix $D(\mathbf{u})$ is unbounded, care is required to guarantee that higher order time stepping schemes inherit this monotonicity.

**1.1. Related Results.** Systems containing an elliptic equation for the pressure coupled with a convection-dominated parabolic equation for the concentration of the solvent appear ubiquitously in reservoir modeling [4, 9]. Existence of the weak solutions to equations (1.1)–(1.3) can be found in [5, 12]; however, under the minimal regularity assumed here uniqueness is not guaranteed.

There exist many works in the literature that introduce numerical approximations of the miscible displacement and prove convergence of the scheme by deriving error estimates between the strong and numerical solutions and by assuming enough regularity for the strong solution. In all cases, the time discretization is a variant of Euler's method that is first order in time. Several discretizations in space have been studied. For instance, in [11, 10, 4], a Galerkin approach for both pressure and concentration is combined with a first order in time discretization. In [16], a sequential backward-difference time-stepping scheme is defined that approximates the pressure by a Galerkin method and the concentration by a combination of a Galerkin method and a method of characteristics. Discontinuous Galerkin methods in space have been analyzed in [8]. Mixed methods combined with finite volume methods for a similar problem are analyzed in [14]. In the references given above, boundedness of the diffusion tensor is assumed, and error estimates are derived under sufficient smoothness of the strong solution. If the diffusion tensor is not assumed to be bounded, a more careful analysis has to be done. In [18], a cut-off operator is employed with a first order in time scheme and discontinuous Galerkin in space. Convergence is obtained by deriving error estimates. In [7], an induction argument is used to simultaneously derive error estimates and $L^\infty$ bounds (needed to control the nonlinear terms) for a mixed method combined with a continuous finite element method and an implicit Euler method.

In practice, the solution is not smooth and it is crucial to be able to prove convergence of the numerical scheme under minimal regularity assumptions upon the data and exact solution. Recently Bartels et. al. [1] addresses this issue by obtaining convergence of the discrete solution by an application of the Aubin-Lions compactness result. The scheme in [1] uses an implicit first order Euler method combined with a mixed method and discontinuous Galerkin method in space; boundedness of the diffusion tensor is not assumed.

The importance of our paper is based on the following:

- Convergence of the numerical solution is obtained under minimal regularity assumptions on the data using a generalization of the Aubin-Lions compactness theorem. The Aubin-Lions theorem is not applicable since the numerical approximations are discontinuities in time.
- High order discontinuous Galerkin time discretizations are employed. To our knowledge, this is the first high order in time scheme analyzed for the miscible displacement problem.
- Diffusion tensor $D(\mathbf{u})$ may be unbounded.
- Coefficients $\mathbf{k}$, $\phi$, $d_m$, $\alpha_l$ and $\alpha_t$ may be discontinuous.

The rest of the section introduces some notation and the weak formulation. Our numerical schemes are introduced in Section 2. Convergence is obtained for the low order in time scheme in Section 3 and for the high order in time scheme in Section 4. Conclusions follow.

**1.2. Notation.** Standard notation is used for the Lebesgue $L^p(\Omega)$ spaces, the Sobolev spaces $W^{k,p}(\Omega)$ and $H^k(\Omega) = W^{k,2}(\Omega)$, and the Bochner spaces $L^2[0,T;W^{k,p}(\Omega)]$ and $H^1[0,T;W^{k,p}(\Omega)]$. The inner-product on $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. Recall that

$$H(\Omega;\mathrm{div}) = \{v \in L^2(\Omega)^d : \quad \mathrm{div}(v) \in L^2(\Omega)\}$$

and it is a Hilbert space with norm

$$\|v\|_{H(\Omega;\mathrm{div})} = \left(\|v\|^2_{L^2(\Omega)} + \|\mathrm{div}(v)\|^2_{L^2(\Omega)}\right)^{1/2},$$

and inner-product

$$(v,w)_{H(\Omega;\mathrm{div})} = (v,w) + (\mathrm{div}(v),\mathrm{div}(w)).$$

If $\Gamma \subset \partial\Omega$, then $(q,u)_\Gamma$ denotes the duality pairing between $q \in H^{-1/2}(\Gamma)$ and $u \in H^{1/2}(\Gamma)$. We write $X \hookrightarrow Y$ to designate the embedding of a normed space $X$ in a normed space $Y$ and if the embedding is compact we write $X \hookrightarrow\!\!\!\rightarrow Y$. The dual of a Banach space $X$ is denoted as $X'$.

We recall that if $\mathcal{U} \subset L^p[\theta, T-\theta; H]$ is (pre) compact for all $\theta \in (0,1)$ and bounded in $L^r[0,T;H]$ with $r > p$, then it is (pre) compact in $L^q[0,T;H]$ for all $1 \le q < r$. If

$B_0 \hookrightarrow B_1 \hookrightarrow B_2$ are Banach spaces with $\|u\|_{B_1} \leq M\|u\|_{B_0}^{\theta}\|u\|_{B_2}^{1-\theta}$ for some $\theta \in (0,1)$, then a bounded subset of $B_0$ which is compact in $B_2$ is also compact in $B_1$.

For any subset $\mathcal{O} \subset \mathbb{R}^d$, the space $\mathcal{P}_\ell(\mathcal{O})$ is the space of polynomials of degree less than or equal to $\ell$ on $\mathcal{O}$. For any real numbers $a < b$ and functional space $H$, the space of continuous functions from $[a,b]$ to $H$ is denoted by $\mathcal{C}[a,b;H]$, and $\mathcal{P}_\ell[a,b;H]$ is defined as

$$\mathcal{P}_\ell[a,b;H] = \{\sum_{i=0}^{\ell} t^i v_i : \quad t \in [a,b], \quad v_i \in H, \quad i = 0, \ldots, l\}.$$

Throughout the paper, we denote generic positive constants by $M$ and $m$ (to avoid conflicts with the concentration $c = c(x,t)$).

We assume the boundary of the domain is partitioned as follows:

$$\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}, \quad \text{with} \quad \Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset = \Gamma_{\mathrm{in}} \cap \Gamma_{\mathrm{out}}.$$

The vector $\mathbf{n}$ denotes a unit normal vector outward to $\partial\Omega$. Dirichlet and Neumann conditions for the pressure are imposed on $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{N}}$ respectively;

$$\varphi = \varphi_{\mathrm{D}}, \quad \text{on} \quad \Gamma_{\mathrm{D}}, \qquad \text{and} \qquad \mathbf{u}\cdot\mathbf{n} = \mathbf{u}_{\mathrm{N}}\cdot\mathbf{n}, \quad \text{on} \quad \Gamma_{\mathrm{N}}.$$

If $|\Gamma_{\mathrm{D}}| = 0$, then compatibility conditions on the data and pressure are required;

$$\int_{\partial\Omega} \mathbf{u}_{\mathrm{N}}\cdot\mathbf{n} = \int_\Omega f, \qquad \text{and} \qquad \int_\Omega \varphi = 0.$$

These conditions are implicitly assumed below when this is the case. Dirichlet and Robin conditions for the concentration are imposed upon $\Gamma_{\mathrm{in}}$ and $\Gamma_{\mathrm{out}}$ respectively;

$$c = c_{\mathrm{in}} \quad \text{on } \Gamma_{\mathrm{in}}, \qquad \text{and} \qquad (-c\mathbf{u} + D(\mathbf{u})\nabla c)\cdot\mathbf{n} = q_{\mathrm{out}} \quad \text{on} \quad \Gamma_{\mathrm{out}}.$$

In many situations it is natural to select the boundary partition $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$ for the concentration to correspond to the inflow and outflow portions of the boundary for the velocity field $\mathbf{u}$. If $\Gamma_{\mathrm{N}} \neq \partial\Omega$ the partition becomes time dependent and implicit, and the natural spaces to pose the concentration are also time dependent. This results in an additional layer of technical issues in the analysis, many of which are routine. In order to circumvent these technicalities we assume that the partition $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$ is independent of time, and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{out}} = \emptyset$. The boundary conditions considered here suffice for the common situation where the concentration or its flux are specified in the "far field" where the flow is quiescent.

**1.3. Weak Formulation.** The weak formulation of the problem is posed in the following function spaces.

$$U = \{\mathbf{v} \in H(\Omega; \mathrm{div}) \mid \mathbf{v}\cdot\mathbf{n}|_{\Gamma_{\mathrm{N}}} = 0\}, \qquad P = L^2(\Omega), \qquad C = \{d \in H^1(\Omega) \mid d|_{\Gamma_{\mathrm{in}}} = 0\}.$$

We consider the following weak problem: find $\mathbf{u}-\mathbf{u}_N \in L^\infty(0,T;U)$, $\varphi \in L^\infty(0,T,P)$, and $c - c_{\text{in}} \in L^2(0,T;C)$, such that

$$D(\mathbf{u})^{1/2}\nabla c \in L^2(0,T;L^2(\Omega))$$

and

$$\int_0^T \big( (\mathbf{K}^{-1}(c)\mathbf{u},\mathbf{v}) - (\varphi, \operatorname{div}(\mathbf{v})) \big) = -\int_0^T (\varphi_{\text{D}}, \mathbf{v}\cdot\mathbf{n})_{\Gamma_{\text{D}}}$$
$$\int_0^T (\operatorname{div}(\mathbf{u}),\psi) = \int_0^T (f,\psi)$$

for all $(\mathbf{v},\psi) \in L^1[0,T;U] \times L^1[0,T;P]$, and

$$\int_0^T \big( -(\phi c, d_t) + \big( -c\mathbf{u} + D(\mathbf{u})\nabla c, \nabla d \big)\big) = (\phi c_0, d(0)) + \int_0^T (g,d) + \int_0^T (q_{\text{out}}, d)_{\Gamma_{\text{out}}},$$

for all

$$d \in \left\{ d \in L^4[0,T;C\cap W^{1,4}(\Omega)] \cap H^1[0,T;C'] \;\middle|\; d(T) = 0 \right\}.$$

The condition $d \in L^4[0,T;W^{1,4}(\Omega)]$ on the test function is technical, and is needed since $D(\mathbf{u})$ is not bounded. It will be shown that $D(\mathbf{u})\nabla c$ belongs to $L^{4/3}(\Omega)$ in which case it is natural to require $d \in W^{1,4}(\Omega)$.

Weak solutions will exist when the domain and data are assumed to satisfy the following.

Assumption 1.1. *The domain $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz.*

1. $f \in L^1[0,T;L^\infty(\Omega)] \cap L^2[0,T;L^2(\Omega)]$.
2. $\mathbf{u}_{\text{N}}$ *is the trace of a function in* $L^2[0,T;H(\Omega;\operatorname{div})]$ *with divergence in* $L^1[0,T;L^\infty(\Omega)]$.
3. $\varphi_{\text{D}} \in L^2[0,T;H^{1/2}(\Gamma_{\text{D}})]$.
4. $g \in L^2[0,T;C']$ .
5. $q_{\text{out}} \in L^2[0,T;H^{-1/2}(\Gamma_{\text{out}})]$.
6. $c_{\text{in}}$ *is the trace on* $\Gamma_{\text{in}}$ *of a function in* $L^2[0,T;H^1(\Omega)\cap W^{1,4}(\Omega)]$.
7. $c_0 \in L^2(\Omega)$.

The following structural hypotheses will be assumed for the coefficients.

Assumption 1.2.

1. *There exist constants* $0 < \phi_0 < \phi_1$ *such that the porosity* $\phi \in L^\infty(\Omega)$ *satisfies* $\phi_0 \le \phi(x) \le \phi_1, \quad x \in \Omega$.
2. *The matrix* $\mathbf{K} : \Omega \times \mathbb{R} \to \mathbb{R}^{d\times d}$ *is symmetric, Caratheodory (i.e. measurable in the first argument and continuous in the second), and uniformly bounded and elliptic. In particular, there exist constants* $0 < k_0 < k_1$ *such that*
$$k_0|\xi|^2 \le \xi^T\mathbf{K}(x,\alpha)\xi \le k_1|\xi|^2, \qquad \xi \in \mathbb{R}^d, \quad \alpha \in \mathbb{R}, \qquad x \in \Omega.$$
3. *The coefficients in the matrix* $D(\mathbf{u})$ *of (1.4) satisfy*
$$d_0 \le d_m(x) \le d_1, \qquad 0 \le \alpha_l(x), \alpha_t(x) \le d_1, \qquad x \in \Omega$$
*for some constants* $0 < d_0 < d_1$. *The matrix* $D(\mathbf{u})$ *is not assumed to be bounded.*

*Remark:* In the case $\mathbf{u}_{\mathrm{N}} = 0$ and $c_{\mathrm{in}} = 0$, one can show that the weak solutions

$$c \in L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)] \quad \text{and} \quad D(\mathbf{u})^{1/2}\nabla c \in L^2[0,T;L^2(\Omega)], \tag{1.5}$$

and that they are bounded by a constant that depends on $\|c_0\|_{L^2(\Omega)}$, $\|f\|_{L^1[0,T;L^\infty(\Omega)]}$, $\|g\|_{L^2[0,T;L^2(\Omega)]}$, $\|q_{\mathrm{out}}\|_{L^2[0,T;H^{-1/2}(\Omega)]}$, and the constants $d_0, d_1$. The concentration can be bounded in $L^\infty(\Omega)$ using the maximum principle [5]; however, the numerical solution obtained with higher order numerical schemes will not inherit such bounds. This makes the analysis challenging.

The following lemma shows that the diffusion matrix $D(\mathbf{u})$ and its square root are Lipschitz continuous in $\mathbf{u}$. We remind the reader that $E(\mathbf{u}) = \mathbf{u}^T\mathbf{y}/|\mathbf{u}|^2$.

LEMMA 1.3. *Let $D(\mathbf{u})$ be defined by (1.4). Then*

- *the matrix $D(\mathbf{u})$ is symmetric and positive definite with eigenvalues in the interval*

$$[d_m + |\mathbf{u}|\min(\alpha_l,\alpha_t), d_m + |\mathbf{u}|\max(\alpha_l,\alpha_t)].$$

- *the function $D(.)$ is Lipschitz with constant of the form $M(\alpha_t + |\alpha_l - \alpha_t|)$.*
- *the positive square root of $D(\mathbf{u})$ is*

$$D^{1/2}(\mathbf{u}) = (d_m + \alpha_l|\mathbf{u}|)^{1/2} E(\mathbf{u}) + (d_m + \alpha_t|\mathbf{u}|)^{1/2}(I - E(\mathbf{u})),$$

  *which is Lipschitz with constant of the form $M(\alpha_l + \alpha_t)(d_m^{-1/2} + d_m^{-3/2})$.*
- *there is a constant $M > 0$ that only depends on $d_m, \alpha_l, \alpha_t$ such that*

$$\xi^T D(\mathbf{u})\xi \le M(1 + |\mathbf{u}|)|\xi|^2, \quad \xi \in \mathbb{R}^d. \tag{1.6}$$

*Proof.* For brevity we write $E$ for $E(\mathbf{u}) = \mathbf{u}\mathbf{u}^T/|\mathbf{u}|^2$. By construction $E$ and $(I - E)$ are projection matrices, so they are non-negative definite, symmetric, and have unit norms. Thus, for all $\mathbf{v} \in \mathbb{R}^d$

$$(d_m + |\mathbf{u}|\min(\alpha_l,\alpha_t))\mathbf{v}^T\mathbf{v} \le \mathbf{v}^T D(\mathbf{u})\mathbf{v} \le (d_m + |\mathbf{u}|\max(\alpha_l,\alpha_t))\mathbf{v}^T\mathbf{v}. \tag{1.7}$$

To verify that $D(\mathbf{u})$ is Lipschitz, write

$$D(\mathbf{u}) = (d_m + \alpha_t|\mathbf{u}|)\, I + (\alpha_l - \alpha_t)\,|\mathbf{u}|E = (d_m + \alpha_t|\mathbf{u}|)\, I + (\alpha_l - \alpha_t)\,(\mathbf{u}\mathbf{u}^T)/|\mathbf{u}|.$$

Let $\delta_{ik}$ denote the Kronecker delta. The calculations

$$\frac{\partial}{\partial u_k}|\mathbf{u}| = \frac{u_k}{|\mathbf{u}|}, \qquad \text{and} \qquad \frac{\partial}{\partial u_k}\left(\frac{u_i u_j}{|\mathbf{u}|}\right) = \frac{\delta_{ik}u_j}{|\mathbf{u}|} + \frac{u_i\delta_{jk}}{|\mathbf{u}|} - \frac{u_i u_j u_k}{|\mathbf{u}|^3},$$

show that the derivative of $D(\mathbf{u})$ is bounded, so $|D(\mathbf{u}) - D(\mathbf{v})| \le M|\mathbf{u} - \mathbf{v}|$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Writing $I = (I - E) + E$ shows

$$D(\mathbf{u}) = (d_m + \alpha_l|\mathbf{u}|)E + (d_m + \alpha_t|\mathbf{u}|)(I - E).$$

Since $E^2 = E$, $(I-E)^2 = I - E$ and $(I-E)E = 0$, direct computation shows

$$D^{1/2}(\mathbf{u}) = (d_m + \alpha_l|\mathbf{u}|)^{1/2}E + (d_m + \alpha_t|\mathbf{u}|)^{1/2}(I-E).$$

To verify that this is a Lipschitz function of $\mathbf{u}$, write

$$\begin{aligned} D^{1/2}(\mathbf{u}) &= (d_m + \alpha_t|\mathbf{u}|)^{1/2}I + \left((d_m + \alpha_l|\mathbf{u}|)^{1/2} - (d_m + \alpha_t|\mathbf{u}|)^{1/2}\right)E \\ &= (d_m + \alpha_t|\mathbf{u}|)^{1/2}I + \left(\frac{(\alpha_l - \alpha_t)}{(d_m + \alpha_l|\mathbf{u}|)^{1/2} + (d_m + \alpha_t|\mathbf{u}|)^{1/2}}\right)|\mathbf{u}|E. \end{aligned}$$

Since the derivative

$$\frac{\partial}{\partial u_k}(d_m + \alpha|\mathbf{u}|)^{1/2} = \frac{\alpha}{2(d_m + \alpha|\mathbf{u}|)^{1/2}}\frac{u_k}{|\mathbf{u}|}$$

is bounded, it follows that the coefficients of $I$ and $|\mathbf{u}|E = \mathbf{u}\mathbf{u}^T/|\mathbf{u}|$ in the formula for $D^{1/2}(\mathbf{u})$, and hence $D(\mathbf{u})^{1/2}$ itself, are Lipschitz. The result (1.6) is immediate. $\square$

**2. Numerical Scheme.** Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\Omega$ and let $RT_k(\mathcal{T}_h)$ denote the space of Raviart-Thomas elements of order $k \geq 0$, i.e.

$$RT_k(\mathcal{T}_h) = \{\mathbf{u} \in H(\text{div};\Omega) \mid \mathbf{u}|_K \in \mathcal{P}_k(K) + x\mathcal{P}_k(K),\ K \in \mathcal{T}_h\}.$$

Define the finite-dimensional subspaces

$$\begin{aligned} U_h &= RT_k(\mathcal{T}_h) \cap U \\ P_h &= \{\psi_h \in P \mid \psi_h|_K \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h\} \\ C_h &= \{c_h \in C \mid c_h|_K \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h\}. \end{aligned}$$

While it is not necessary to let $C_h$ have the same degree polynomials as the other two spaces, there is no reason not to. Also, for definiteness we assume $\mathcal{T}_h$ is a simplicial mesh; however, any of the classical mixed finite element spaces suffices. For example, on a quadrilateral mesh the $\text{BDFM}_k(\mathcal{T}_h)$ spaces for the velocity and corresponding tensor product spaces for the pressure and concentration may be utilized [2].

Let $0 = t^0 < t^1 < \ldots < t^N = T$ be a partition, and let $\Delta t = \max_{i=1,\ldots,N}(t^i - t^{i-1})$.

The following assumptions are made for the analysis of the numerical scheme.

Assumption 2.1.

1. *The partition is quasi-uniform, i.e. there exists $\theta \in (0,1]$ such that*

$$\theta \Delta t \leq \min_{1\leq n\leq N}(t^n - t^{n-1}). \tag{2.1}$$

2. *To simplify the exposition it is assumed that the Dirichlet data is homogeneous; namely, $\mathbf{u}_{\text{N}} = 0$ and $c_{\text{in}} = 0$.*
3. *We assume that the partition $\Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ is independent of time, and $\Gamma_{\text{D}} \cap \Gamma_{\text{out}} = \emptyset$.*

4. *The approximation of the initial concentration $c_0$ is denoted by $c_{h-}^0$ and it is equal to the orthogonal projection of $c_0$ in $C_h$ with respect to a weighted $L^2$ inner product:*

$$\forall d_h \in C_h, \quad (c_{h-}^0, \phi d_h) = (c_0, \phi d_h).$$

REMARK 2.2. *Nonhomogeneous data may be accommodated using the usual translation argument provided* $\mathrm{div}(\mathbf{u}_N)$ *and $f$ have the same regularity, and*

$$c_{\mathrm{in}} \in H^1[0,T; H_\phi^{-1}(\Omega)] \cap L^\infty[0,T; W^{1,4}(\Omega)], \qquad \text{where} \qquad \|c_{\mathrm{in}}\|_{H_\phi^{-1}(\Omega)} = \sup_{d \in H^1(\Omega)} \frac{(\phi c_{\mathrm{in}}, d)_{L^2(\Omega)}}{\|d\|_{H^1(\Omega)}}$$

*is the dual norm when the pivot space is taken to have the weighted $L^2(\Omega)$ inner product.*

Fix an integer $\ell \geq 0$. For any function $d$ such that $d|_{[t^{n-1},t^n]} \in \mathcal{P}_\ell[t^{n-1}, t^n, C]$, define

$$d_-^n = \lim_{\epsilon \downarrow 0} d(t_n - \epsilon, \cdot), \quad d_+^n = \lim_{\epsilon \downarrow 0} d(t_n + \epsilon, \cdot), \quad [d^n] = d_+^n - d_-^n.$$

We consider approximate solutions of equations (1.1)-(1.3) satisfying

$$\mathbf{u}_h \in \mathcal{P}_\ell[t^{n-1}, t^n, U_h], \qquad \varphi_h \in \mathcal{P}_\ell[t^{n-1}, t^n, P_h], \qquad c_h \in \mathcal{P}_\ell[t^{n-1}, t^n, C_h],$$

and

$$\int_{t^{n-1}}^{t^n} \big( (\mathbf{K}^{-1}(c_h)\mathbf{u}_h, \mathbf{v}_h) - (\varphi_h, \mathrm{div}(\mathbf{v}_h)) \big) = -\int_{t^{n-1}}^{t^n} (\varphi_{\mathrm{D}}, \mathbf{v}_h \cdot \mathbf{n})_{\Gamma_{\mathrm{D}}}, \tag{2.2}$$

$$\int_{t^{n-1}}^{t^n} (\mathrm{div}(\mathbf{u}_h), \psi_h) = \int_{t^{n-1}}^{t^n} (f, \psi_h), \tag{2.3}$$

and for $\ell = 0$ or $\ell = 1$

$$\int_{t^{n-1}}^{t^n} \big( (\phi c_{ht}, d_h) + (-c_h \mathbf{u}_h + D(\mathbf{u}_h)\nabla c_h, \nabla d_h) \big) + ([c_h^{n-1}], \phi d_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} \big( (g, d_h) + (q_{\mathrm{out}}, d_h)_{\Gamma_{\mathrm{out}}} \big), \tag{2.4}$$

for all $\mathbf{v}_h \in \mathcal{P}_\ell[t^{n-1}, t^n, U_h]$, $\psi_h \in \mathcal{P}_\ell[t^{n-1}, t^n, P_h]$, $d_h \in \mathcal{P}_\ell[t^{n-1}, t^n, C_h]$. Equation (2.4) is used for the low order in time schemes ($\ell \leq 1$).

For the high order scheme ($\ell > 1$), a quadrature rule $Q^n : \mathcal{C}[t^{n-1}, t^n] \to \mathbb{R}$ is used to evaluate the nonlinear term and equation (2.4) is replaced by

$$\int_{t^{n-1}}^{t^n} (\phi c_{ht}, d_h) + Q^n \big( (-c_h \mathbf{u}_h + D(\mathbf{u}_h)\nabla c_h, \nabla d_h) \big) + ([c_h^{n-1}], \phi d_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} \big( (g, d_h) + (q_{\mathrm{out}}, d_h)_{\Gamma_{\mathrm{out}}} \big).$$

The construction of a quadrature rule which preserves the formal high order accuracy and inherits the stability of the continuous problem is given in Section 4 below.

Existence of solutions to the discrete problem with sufficiently small time steps follows from Brouwer's fixed point theorem [17, Proposition II.2.1] and the stability estimates established below.

**3. Convergence of the Numerical Scheme.** In this section convergence of the numerical scheme is established. The first step is to show that the numerical schemes inherit the natural energy estimates of the underlying equations. Low order time stepping schemes inherit these estimates directly and the higher order schemes which use quadrature to evaluate the nonlinear terms are also stable. The second step is to establish sufficient compactness of the concentrations to allow passage to the limit in the nonlinear terms. Our main results are Theorem 3.10 and Theorem 3.11. The modifications required to establish stability and compactness of the concentrations for the higher order schemes are technical but routine, so for this reason are postponed until Section 4 (see Theorem 4.4).

Throughout this section it is assumed that the discrete solutions are computed using a quasi-regular family of triangulations of $\Omega$ and that the temporal partition satisfies (2.1).

**3.1. Stability of the Pressure and the Velocity.** In this section energy estimates are used to establish stability of the numerical scheme (2.2)-(2.4). Bounds on the discrete pressure and velocity can be established independently of the concentration. Stability of the low order ($\ell \leq 1$) DG time stepping scheme for the concentration then follow.

LEMMA 3.1. *There exists a constant $M > 0$ independent of $h$ and $\Delta t$ such that solutions of the numerical scheme (2.2)-(2.3) satisfy the following bounds.*

1. *If $1 \leq p, q \leq \infty$ and $f \in L^p[0,T; L^q(\Omega)]$, then*

$$\|\mathrm{div}(\mathbf{u}_h)\|_{L^p[0,T;L^q(\Omega)]} \leq M \|f\|_{L^p[0,T;L^q(\Omega)]}.$$

2. *If $1 \leq p \leq \infty$, $f \in L^p[0,T; L^2(\Omega)]$, and $\varphi_{\mathrm{D}} \in L^p[0,T; H^{1/2}(\Gamma_{\mathrm{D}})]$, then*

$$\|\mathbf{u}_h\|_{L^p[0,T;H(\Omega;\mathrm{div})]} + \|\varphi_h\|_{L^p[0,T;L^2(\Omega)]} \leq M \left(\|f\|_{L^p[0,T;L^2(\Omega)]} + \|\varphi_{\mathrm{D}}\|_{L^p[0,T;H^{1/2}(\Gamma_{\mathrm{D}})]}\right).$$

The proof of the lemma uses the following two well known properties of the discrete spaces being utilized [3, 15]. Recall that if $\Gamma_{\mathrm{N}} = \partial\Omega$ then we assume additionally that the average of functions in $P_h$ vanish.

LEMMA 3.2. *There exists a constant $m > 0$ depending only upon $\Omega$ such that*

$$\sup_{\mathbf{u}_h \in U_h} \frac{\int_\Omega p_h \,\mathrm{div}(\mathbf{u}_h)}{\|\mathbf{u}_h\|_{H(\Omega;\mathrm{div})}} \geq m \|p_h\|_{L^2(\Omega)}, \quad p_h \in P_h.$$

*In particular, if $Z_h = \{\mathbf{u}_h \in U_h \mid \mathrm{div}(\mathbf{u}_h) = 0\}$ and $U_h = Z_h \oplus Z_h^\perp$ is the orthogonal decomposition, then there exists a linear operator $L_h : P_h \to Z_h^\perp$ with $\|L_h\|_{\mathcal{L}(P_h,U_h)} \leq 1$ such that*

$$m\|p_h\|^2_{L^2(\Omega)} \leq \int_\Omega p_h \,\mathrm{div}(L_h(p_h)), \quad p_h \in P_h,$$

*and if $\mathbf{u}_h \in Z_h^\perp$ then $m\|\mathbf{u}_h\|_{H(\Omega;\mathrm{div})} \leq \|\mathrm{div}(\mathbf{u}_h)\|_{L^2(\Omega)}$.*

The following lemma follows from an elementary scaling (parent element) calculation.

Lemma 3.3. *Let $V$ be a linear space and $(.,.)_V$ be a (semi) inner product on $V$; $w \geq 0$ be a non-zero element of $L^1(0,1)$; and $0 < a < b$. Then there exists a constant $M_\ell > 0$, depending only upon $\ell$ and $w$, such that for all $u \in \mathcal{P}_\ell[a,b;V]$*

$$\|u\|_{L^p[a,b;V]} \leq (b-a)^{1/p-1/2} \left( M_\ell \int_a^b w((t-a)/(b-a)) \|u(t)\|_V^2 \, dt \right)^{1/2}, \quad 1 \leq p \leq \infty.$$

*In particular, if $1/p + 1/p' = 1$ then*

$$\|u\|_{L^p[a,b;V]} \|u\|_{L^{p'}[a,b;V]} \leq M_\ell \int_a^b w((t-a)/(b-a)) \|u(t)\|_V^2 \, dt.$$

*Proof of Lemma 3.1*: For each $K \in \mathcal{T}_h$ and $1 \leq n \leq N$ let $\Pi_h : L^2((t^{n-1}, t^n) \times K) \to \mathcal{P}_\ell[t^{n-1}, t^n, \mathcal{P}_k(K)]$ denote the $L^2$ projection. A parent element calculation shows that there exists a constant $M > 0$ depending only upon the parent element such that

$$\|\Pi_h f\|_{L^p[t^{n-1},t^n;L^q(K)]} \leq M \|f\|_{L^p[t^{n-1},t^n;L^q(K)]}, \quad 1 \leq p, q \leq \infty. \tag{3.1}$$

Since $\mathrm{div}(\mathbf{u}_h) \in P_h$ it follows from (2.3) that

$$\mathrm{div}(\mathbf{u}_h) = \Pi_h(f), \tag{3.2}$$

and the first estimate follows.

Next, let $Z_h \subset U_h$ be the kernel of the discrete divergence introduced in Lemma 3.2 and let $U_h = Z_h \oplus Z_h^\perp$ denote the orthogonal decomposition. Let $\mathbf{u}_h = \mathbf{z}_h + \mathbf{u}_h^\perp$ be the decomposition of $\mathbf{u}_h$. Since the decomposition is independent of time it follows that each of $\mathbf{z}_h$ and $\mathbf{u}_h^\perp$ are in $\mathcal{P}_\ell[t^{n-1}, t^n, U_h]$. From Lemma 3.2 we find

$$M \|\mathbf{u}_h^\perp\|_{H(\Omega;\mathrm{div})} \leq \|\mathrm{div}(\mathbf{u}_h^\perp)\|_{L^2(\Omega)} = \|\mathrm{div}(\mathbf{u}_h)\|_{L^2(\Omega)},$$

and since $\mathrm{div}(\mathbf{u}_h) = \Pi_h(f)$ it follows that

$$\|\mathbf{u}_h^\perp\|_{L^p[t^{n-1},t^n;H(\Omega;\mathrm{div})]} \leq M \|\mathrm{div}(\mathbf{u}_h)\|_{L^p[t^{n-1},t^n;L^2(\Omega)]} \leq M \|f\|_{L^p[t^{n-1},t^n;L^2(\Omega)]}. \tag{3.3}$$

To estimate $\mathbf{z}_h$ select it to be the test function in equation (2.2) to get

$$\int_{t^{n-1}}^{t^n} (\mathbf{K}^{-1}(c_h)(\mathbf{z}_h + \mathbf{u}_h^\perp), \mathbf{z}_h) = \int_{t^{n-1}}^{t^n} (\mathbf{K}^{-1}(c_h)\mathbf{u}_h, \mathbf{z}_h) = \int_{t^{n-1}}^{t^n} -(\varphi_\mathrm{D}, \mathbf{z}_h \cdot \mathbf{n})_{\Gamma_\mathrm{D}}.$$

Upon recalling that $\|\mathbf{z}_h\|_{H(\Omega;\mathrm{div})} = \|\mathbf{z}_h\|_{L^2(\Omega)}$ and the assumptions on $\mathbf{K}$, it follows that

$$\begin{aligned}
\|\mathbf{z}_h\|^2_{L^2[t^{n-1},t^n;H(\Omega;\mathrm{div})]} &\leq M \int_{t^{n-1}}^{t^n} (\mathbf{K}^{-1}(c_h)\mathbf{z}_h, \mathbf{z}_h) \\
&\leq M \int_{t^{n-1}}^{t^n} \left| (\mathbf{K}^{-1}(c_h)\mathbf{u}_h^\perp, \mathbf{z}_h) + (\varphi_\mathrm{D}, \mathbf{z}_h.\mathbf{n})_{\Gamma_\mathrm{D}} \right| \\
&\leq M \|\mathbf{z}_h\|_{L^{p'}[t^{n-1},t^n;H(\Omega;\mathrm{div})]} \left( \|\mathbf{u}_h^\perp\|_{L^p[t^{n-1},t^n;L^2(\Omega)]} + \|\varphi_\mathrm{D}\|_{L^p[t^{n-1},t^n;H^{1/2}(\Gamma_\mathrm{D})]} \right),
\end{aligned}$$

where the trace theorem on $H(\Omega;\mathrm{div})$ was used in the last line and $1/p + 1/p' = 1$. The bound (3.3) and Lemma 3.3 (with weight $w \equiv 1$) then show

$$\|\mathbf{z}_h\|_{L^p[t^{n-1},t^n;H(\Omega;\mathrm{div})]} \leq M \left( \|f\|_{L^p[t^{n-1},t^n;L^2(\Omega)]} + \|\varphi_{\mathrm{D}}\|_{L^p[t^{n-1},t^n;H^{1/2}(\Gamma_{\mathrm{D}})]} \right),$$

from which the bound on $\|\mathbf{u}_h\|_{L^p[t^{n-1},t^n;H(\Omega;\mathrm{div})]}$ follows.

Since the operator $L_h : P_h \to Z_h^\perp$ in Lemma 3.2 is independent of time, it follows that $L_h(\varphi_h) \in \mathcal{P}_\ell[t^{n-1}, t^n, U_h]$. We may then set $\mathbf{v}_h = L_h(\varphi_h)$ in (2.2) to find

$$M \int_{t^{n-1}}^{t^n} \|\varphi_h\|^2_{L^2(\Omega)} \leq \int_{t^{n-1}}^{t^n} (\varphi_h, \mathrm{div}(L_h(\varphi_h))) = \int_{t^{n-1}}^{t^n} \left( (\mathbf{K}^{-1}(c_h)\mathbf{u}_h, L_h(\varphi_h)) + (\varphi_{\mathrm{D}}, L_h(\varphi_h))_{\Gamma_{\mathrm{D}}} \right).$$

Using the trace theorem, the assumptions on $\mathbf{K}$, and Lemma 3.3 it follows that

$$\begin{aligned} \|\varphi_h\|_{L^p[t^{n-1},t^n;L^2(\Omega)]} &\leq M \left( \|\mathbf{u}_h\|_{L^p[t^{n-1},t^n;L^2(\Omega)]} + \|\varphi_{\mathrm{D}}\|_{L^p[t^{n-1},t^n;H^{1/2}(\Gamma_{\mathrm{D}})]} \right) \\ &\leq M \left( \|f\|_{L^p[t^{n-1},t^n;L^2(\Omega)]} + \|\varphi_{\mathrm{D}}\|_{L^p[t^{n-1},t^n;H^{1/2}(\Gamma_{\mathrm{D}})]} \right). \end{aligned}$$

□

**3.2. Stability of the Concentration in the Case $\ell \leq 1$.** Next we derive a priori bounds for the discrete concentration computed using the low order DG time stepping schemes, $\ell = 0$ and $\ell = 1$. The following discrete Gronwall inequality will be required.

LEMMA 3.4. *Let $\{\tau^i\}_{i=1}^N \subset [0, 1)$, and $\{a^i\}_{i=0}^N$, $\{b^i\}_{i=1}^N$, and $\{f^i\}_{i=1}^N$ be subsets of $[0, \infty)$. If*

$$(1 - \tau^n)a^n + b^n \leq a^{n-1} + f^n, \qquad n = 1, 2, \ldots N,$$

*then*

$$a^N + \sum_{n=1}^N \frac{b^n}{\prod_{i=n}^N (1 - \tau^i)} \leq \frac{a^0}{\prod_{i=1}^N (1 - \tau^i)} + \sum_{n=1}^N \frac{f^n}{\prod_{i=n}^N (1 - \tau^i)}.$$

*In particular, if $\max_{1 \leq i \leq N} \tau^i \leq \lambda < 1$ and $t^N \equiv \sum_{i=1}^N \tau_i$ then*

$$a^N + \sum_{n=1}^N b^n \leq \frac{e^{t^N}}{1 - \lambda^2} \left( a^0 + \sum_{n=1}^N f^n \right).$$

LEMMA 3.5 (Stability of low order schemes). *Let the data and coefficients satisfy Assumptions 1.1 and 1.2 respectively. Then there exist positive constants $\lambda = \lambda(\|f\|_{L^1[0,T;L^\infty(\Omega)]})$ and $M$, independent of $h$ and $\Delta t$, such that the concentrations computed using the low order DG time stepping schemes, $\ell = 0$ or $\ell = 1$ in (2.4), satisfy*

$$\begin{aligned} \max_{1 \leq n \leq N} \|\phi^{1/2} c^n_{h-}\|^2_{L^2(\Omega)} + \sum_{i=1}^N \|[\phi^{1/2} c_h^{n-1}]\|^2_{L^2(\Omega)} + \int_0^T (D(\mathbf{u}_h)\nabla c_h, \nabla c_h) \\ \leq M \exp(M\|f\|_{L^1[0,t^n,L^\infty(\Omega)]}) \left( \|\phi^{1/2} c_0\|^2_{L^2(\Omega)} + \|g\|^2_{L^2[0,T;C']} + \|q_{\mathrm{out}}\|^2_{L^2(0,T;H^{-1/2}(\Gamma_{\mathrm{out}})} \right), \end{aligned} \tag{3.4}$$

*provided $\lambda\Delta t < 1$. In particular,*

$$\|c_h\|_{L^\infty[0,T;L^2(\Omega)]}, \qquad \|c_h\|_{L^2[0,T;H^1(\Omega)]}, \qquad \text{and} \qquad \|D(\mathbf{u}_h)^{1/2}\nabla c_h\|_{L^2[0,T;L^2(\Omega)]}$$

*are bounded independently of $h$ and $\Delta t$.*

*Proof.* For readability we use the notation $c_-^n = c_{h-}^n$ and similarly $c_+^n = c_{h+}^n$. Set $d_h = c_h$ in equation (2.4) to obtain

$$\frac{1}{2}\|\phi^{1/2}c_-^n\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\phi^{1/2}c_+^{n-1}\|_{L^2(\Omega)}^2 - (c_-^{n-1}, \phi c_+^{n-1}) + \int_{t^{n-1}}^{t^n} (D(\mathbf{u}_h)\nabla c_h, \nabla c_h)$$
$$= \frac{1}{2}\int_{t^{n-1}}^{t^n} \big( - (\operatorname{div}(\mathbf{u}_h), c_h^2) + (\mathbf{u}_h \cdot \mathbf{n}, c_h^2)_{\partial\Omega}\big) + \int_{t^{n-1}}^{t^n} (g, c_h) + \int_{t^{n-1}}^{t^n} (q_{\text{out}}, c_h)_{\Gamma_{\text{out}}}.$$

The first three terms on the left may be rewritten as

$$\|\phi^{1/2}c_-^n\|_{L^2(\Omega)}^2 + \|\phi^{1/2}c_+^{n-1}\|_{L^2(\Omega)}^2 - 2(c_-^{n-1}, \phi c_+^{n-1}) = \|\phi^{1/2}c_-^n\|_{L^2(\Omega)}^2 + \|[\phi^{1/2}c^{n-1}]\|_{L^2(\Omega)}^2 - \|\phi^{1/2}c_-^{n-1}\|_{L^2(\Omega)}^2.$$

Using the fact that $\mathbf{u}_h \cdot \mathbf{n} = 0$ on $\Gamma_{\text{N}}$ and $c_h = 0$ on $\Gamma_{\text{in}} \supset \Gamma_{\text{D}}$ we obtain

$$\frac{1}{2}\|\phi^{1/2}c_-^n\|_{L^2(\Omega)}^2 + \frac{1}{2}\|[\phi^{1/2}c^{n-1}]\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (D(\mathbf{u}_h)\nabla c_h, \nabla c_h)$$
$$= \frac{1}{2}\|\phi^{1/2}c_-^{n-1}\|_{L^2(\Omega)}^2 - \frac{1}{2}\int_{t^{n-1}}^{t^n} (\operatorname{div}(\mathbf{u}_h), c_h^2) + \int_{t^{n-1}}^{t^n} (g, c_h) + \int_{t^{n-1}}^{t^n} (q_{\text{out}}, c_h)_{\Gamma_{\text{out}}}. \tag{3.5}$$

Next, let $\Pi_h : L^2[t^{n-1}, t^n; L^2(\Omega)] \to \mathcal{P}_\ell[t^{n-1}, t^n; P_h]$ be the $L^2$ projection. We may use equation (3.2) to simplify the term involving $\operatorname{div}(\mathbf{u}_h)$,

$$\int_{t^{n-1}}^{t^n} (\operatorname{div}(\mathbf{u}_h), c_h^2) = \int_{t^{n-1}}^{t^n} (\operatorname{div}(\mathbf{u}_h), \Pi_h(c_h^2)) = \int_{t^{n-1}}^{t^n} (\Pi_h(f), \Pi_h(c_h^2)) = \int_{t^{n-1}}^{t^n} (\Pi_h(f), c_h^2).$$

Therefore (3.5) becomes

$$\|\phi^{1/2}c_-^n\|_{L^2(\Omega)}^2 + \|[\phi^{1/2}c^{n-1}]\|_{L^2(\Omega)}^2 + 2\int_{t^{n-1}}^{t^n} (D(\mathbf{u}_h)\nabla c_h, \nabla c_h)$$
$$= \|\phi^{1/2}c_-^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} \big(2(g, c_h) + 2(q_{\text{out}}, c_h)_{\Gamma_{\text{out}}} - (\Pi_h(f), c_h^2)\big)$$
$$\le \|\phi^{1/2}c_-^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} \Big\{ M\|g\|_{C'}\|c_h\|_{H^1(\Omega)}$$
$$+ M\|q_{\text{out}}\|_{H^{-1/2}(\Gamma_{\text{out}})}\|c_h\|_{H^1(\Omega)} + \|\Pi_h(f)\|_{L^\infty(\Omega)}\|c_h\|_{L^2(\Omega)}^2 \Big\}.$$

Equation (1.7) shows $\|\nabla c_h\|_{L^2(\Omega)} \le d_m^{-1/2}\|D(\mathbf{u}_h)^{1/2}\nabla c_h\|_{L^2(\Omega)}$, and using Poincaré's inequality it follows that

$$\|\phi^{1/2}c_-^n\|_{L^2(\Omega)}^2 + \|[\phi^{1/2}c^{n-1}]\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (D(\mathbf{u}_h)\nabla c_h, \nabla c_h) \le \|\phi^{1/2}c_-^{n-1}\|_{L^2(\Omega)}^2$$
$$+ M\left(\|g\|_{L^2[t^{n-1},t^n;C']}^2 + \|q_{\text{out}}\|_{L^2[t^{n-1},t^n;H^{-1/2}(\Gamma_{\text{out}})]}^2\right) + \int_{t^{n-1}}^{t^n} \|\Pi_h(f)\|_{L^\infty(\Omega)}\|c_h\|_{L^2(\Omega)}^2.$$

If $\ell = 0$, then $c_h$ is piecewise constant and equal to $c_-^n$ on the interval $(t^{n-1}, t^n)$ and

$$\int_{t^{n-1}}^{t^n} \|\Pi_h(f)\|_{L^\infty(\Omega)} \|c_h\|_{L^2(\Omega)}^2 = \|c_-^n\|_{L^2(\Omega)}^2 \int_{t^{n-1}}^{t^n} \|\Pi_h(f)\|_{L^\infty(\Omega)} = M\|c_-^n\|_{L^2(\Omega)}^2 \int_{t^{n-1}}^{t^n} \|f\|_{L^\infty(\Omega)}.$$

When $\ell = 1$ write

$$c_h(t)|_{[t^{n-1},t^n]} = \frac{t - t^{n-1}}{t^n - t^{n-1}} c_-^n + \frac{t^n - t}{t^n - t^{n-1}} c_+^{n-1}.$$

Then

$$\|c_h\|_{L^2(\Omega)}^2 \le \frac{t - t^{n-1}}{t^n - t^{n-1}} \|c_-^n\|_{L^2(\Omega)}^2 + \frac{t^n - t}{t^n - t^{n-1}} \|c_+^{n-1}\|_{L^2(\Omega)}^2$$

and

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|\Pi_h(f)\|_{L^\infty(\Omega)} \|c_h\|_{L^2(\Omega)}^2 &\le M \left( \|c_-^n\|_{L^2(\Omega)}^2 + \|c_+^{n-1}\|_{L^2(\Omega)}^2 \right) \int_{t^{n-1}}^{t^n} \|f\|_{L^\infty(\Omega)} \\ &\le M \left( \|c_-^n\|_{L^2(\Omega)}^2 + \|[c^{n-1}]\|_{L^2(\Omega)}^2 + \|c_-^{n-1}\|_{L^2(\Omega)}^2 \right) \int_{t^{n-1}}^{t^n} \|f\|_{L^\infty(\Omega)}.\end{aligned}$$

Since $\|c_h\|_{L^2(\Omega)}^2 \le (1/\phi_0)\|\phi^{1/2} c_h\|_{L^2(\Omega)}^2$, it follows that if $\ell = 0$ or $\ell = 1$ then

$$\begin{aligned}(1 - \lambda^n)\|\phi^{1/2} c_-^n\|_{L^2(\Omega)}^2 + (1 - \lambda^n)\|[\phi^{1/2} c^{n-1}]\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (D(\mathbf{u}_h)\nabla c_h, \nabla c_h) \\ \le (1 + \lambda^n)\|\phi^{1/2} c_-^{n-1}\|_{L^2(\Omega)}^2 + M\big(\|g\|_{L^2(t^{n-1},t^n;C')}^2 + \|q_{\text{out}}\|_{L^2(t^{n-1},t^n;H^{-1/2}(\Gamma_{\text{out}}))}^2\big)\end{aligned}$$

where

$$\lambda^n = (M/\phi_0) \int_{t^{n-1}}^{t^n} \|f\|_{L^\infty(\Omega)}.$$

Since $f \in L^1[0, T; L^\infty(\Omega)]$ it follows that there exists $\lambda > 0$ such that $|t - s| < 1/\lambda$ implies

$$\int_s^t \|f\|_{L^\infty(\Omega)} \le \phi_0/2M.$$

In particular, $\lambda^n < 1/2$ when $\lambda \Delta t < 1$ and application of the discrete Gronwall inequality establishes the Lemma. □

**3.3. Compactness of the Concentration.** Compactness of solutions to evolution equations is frequently established using the Lions Aubin Theorem [17]. However, this theorem is not applicable to discontinuous solutions since their time derivatives are not integrable. To circumvent this difficulty we will use the following theorem.

THEOREM 3.6. *Let $H$ be a Hilbert space with inner-product $(\cdot,\cdot)_H$ and $V$ and $W$ be Banach spaces equipped with norms $\|\cdot\|_V$ and $\|\cdot\|_W$. Assume that*

$$W \hookrightarrow V \hookrightarrow\!\!\!\!\rightarrow H \hookrightarrow W'$$

*are dense embeddings with $V$ compactly embedded in $H$. Let $\ell \geq 0$ be an integer and $h > 0$ be a (mesh) parameter. For each $h$, let $W_h \subset W$ be a closed subspace and $0 = t_h^0 < t_h^1 < \ldots < t_h^N = T$ be a uniform partition of $[0, T]$. Let $\Pi_h : H \to W_h$ denote the orthogonal projection, and assume that its restriction to $W$ is stable in the sense that there exists a constant $M > 0$ independent of $h$ such that $\|\Pi_h w\|_W \leq M\|w\|_W$ for $w \in W$.*

*Fix $1 < p < \infty$, $1 \leq q < \infty$ with $1/p + 1/q \geq 1$ and assume that*

1. *For each $h > 0$, $w_h \in \{w_h \in L^p[0, T; W] \mid w_h|_{(t_h^{n-1}, t^n)_h} \in \mathcal{P}_\ell(t_h^{n-1}, t_h^n; W_h)\}$ and on each interval satisfies*
$$\forall z_h \in \mathcal{P}_\ell(t_h^{n-1}, t_h^n; W_h), \quad \int_{t_h^{n-1}}^{t_h^n} (w_{ht}, z_h)_H + (w_{h+}^{n-1} - w_{h-}^{n-1}, z_{h+}^{n-1})_H = \int_{t_h^{n-1}}^{t_h^n} F_h(z_h).$$
2. *The sequence $\{w_h\}_{h>0}$ is bounded in $L^p[0, T; V]$.*
3. *For each $h > 0$, $F_h \in L^q[0, T; W_h']$ and $\{\|F_h\|_{L^q[0,T;W_h']}\}_{h>0} \subset \mathbb{R}$ bounded.*

*Then the set $\{w_h\}_{h>0}$ is precompact in $L^p[0, T; H] \cap L^r[0, T; W']$ for each $1 \leq r < \infty$.*

**Remarks:**

1. This theorem is a variation of [19, Theorem 3.1] and the proof is sketched at the end of this section.
2. The restriction in [19] to uniform partitions of $[0, T]$ was made to simplify the proof and can be relaxed to "quasi uniform" as defined in (2.1).
3. The requirement that $F_h \in L^q[0, T; W_h']$ is not sharp since the DG time stepping scheme only requires $F_h$ to act on piecewise polynomials.

We now establish compactness of the numerical approximations of solutions to the concentration equation.

THEOREM 3.7. *Let the data and coefficients satisfy Assumptions 1.1 and 1.2 respectively and suppose that the maximal time step $\Delta t$ tends to zero with the mesh parameter. Then the concentrations $\{c_h\}_{h>0}$ computed using (2.2)–(2.4) with $\ell = 0$ or $\ell = 1$ are precompact in $L^2[0, T; L^2(\Omega)] \cap L^r[0, T; (C \cap W^{1,4}(\Omega))']$ for any $1 \leq r < \infty$.*

*Proof.* We apply Theorem 3.6 with parameters $p = 2$, $q = 1$, and spaces $H = L^2(\Omega)$, $V = C \subset H^1(\Omega)$, $W = C \cap W^{1,4}(\Omega)$ and $W_h = C_h$. To verify the assumptions of the theorem, let the weighted inner-product on $H = L^2(\Omega)$ be $(v, d)_H = (\phi v, d)$ and define $F_h \in L^1[0, T; W_h']$ by

$$\forall d_h \in \mathcal{P}_\ell[t^{n-1}, t^n, C_h], \quad F_h(d_h) = (g, d_h) + (c_h \mathbf{u}_h - D(\mathbf{u}_h)\nabla c_h, \nabla d_h) + (q_{\text{out}}, d_h)_{\Gamma_{\text{out}}}.$$

With this notation, the discrete weak statement (2.4) for the concentration takes the form assumed in Theorem 3.6;

$$\int_{t^{n-1}}^{t^n} (c_{ht}, d_h)_H + ([c_h^{n-1}], d_{h+}^{n-1})_H = \int_{t^{n-1}}^{t^n} F_h(d_h).$$

Lemma 3.5 shows that Assumption 2 of Theorem 3.6 is satisfied. To establish the third assumption of the theorem we show each term in the definition of $F_h$ is bounded in

$L^1[0,T;W_h']$. The terms containing the data are bounded as

$$\int_0^T |(g,d_h)+(q_{\rm out},d_h)_{\Gamma_{\rm out}}| \le (\|g\|_{L^1[0,T;C']}+\|q_{\rm out}\|_{L^1(0,T;H^{-1/2}(\Gamma_{\rm out}))})\|d_h\|_{L^\infty[0,T;H^1(\Omega)]}.$$

The second term in the definition of $F_h$ is bounded using the Sobolev embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$,

$$\begin{aligned}\int_0^T (c_h\mathbf{u}_h,\nabla d_h) &\le \int_0^T \|c_h\|_{L^4(\Omega)}\|\mathbf{u}_h\|_{L^2(\Omega)}\|\nabla d_h\|_{L^4(\Omega)}\\ &\le M\int_0^T \|c_h\|_{H^1(\Omega)}\|\mathbf{u}_h\|_{L^2(\Omega)}\|\nabla d_h\|_{L^4(\Omega)}\\ &\le M\|c_h\|_{L^2[0,T;H^1(\Omega)]}\|\mathbf{u}_h\|_{L^2[0,T;L^2(\Omega)]}\|\nabla d_h\|_{L^\infty[0,T;L^4(\Omega)]},\end{aligned}$$

and the third term is bounded by using (1.6),

$$\begin{aligned}\int_0^T (D(\mathbf{u}_h)\nabla c_h,\nabla d_h) &\le \int_0^T \|D^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2(\Omega)}\|D^{1/2}(\mathbf{u}_h)\nabla d_h\|_{L^2(\Omega)}\\ &\le M\int_0^T \|D^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2(\Omega)}\left(\|\nabla d_h\|_{L^2(\Omega)}+\|\mathbf{u}_h\|^{1/2}_{L^2(\Omega)}\|\nabla d_h\|_{L^4(\Omega)}\right)\\ &\le M\|D^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2[0,T;L^2(\Omega)]}\times\\ &\quad\left(\|\nabla d_h\|_{L^2[0,T;L^2(\Omega)]}+\|\mathbf{u}_h\|^{1/2}_{L^2[0,T;L^2(\Omega)]}\|\nabla d_h\|_{L^4[0,T;L^4(\Omega)]}\right).\end{aligned}$$

Then

$$\begin{aligned}\|F_h\|_{L^1[0,T;W_h']} \le M\big(&\|g\|_{L^1[0,T;C']}+\|\mathbf{u}_h\|_{L^2[0,T;L^2(\Omega)]}\|c_h\|_{L^2[0,T;H^1(\Omega)]}\\ &+\|D^{1/2}(\mathbf{u}_h)\nabla c_h\|_{L^2[0,T;L^2(\Omega)]}(1+\|\mathbf{u}_h\|^{1/2}_{L^2[0,T;L^2(\Omega)]})+\|q_{\rm out}\|_{L^1[0,T;H^{-1/2}(\Gamma_{\rm out})]}\big).\end{aligned}$$

This result, combined with Lemma 3.1 and Lemma 3.5, establishes Assumption 3 of Theorem 3.6. The theorem then shows that the concentrations $\{c_h\}_{h>0}$ are precompact in $L^2(0,T;L^2(\Omega))\cap L^r(0,T;W')$ for each $1\le r<\infty$. $\square$

REMARK 3.8. *Since $W^{1,4}(\Omega)\hookrightarrow H^s(\Omega)$ for $s=1+d/4$ it follows that $(C\cap W^{1,4}(\Omega))'\hookrightarrow H^{-s}$. Interpolating between the inclusions $H^1(\Omega)\hookrightarrow L^2(\Omega)\equiv H^0(\Omega)\hookrightarrow H^{-s}$ then shows*

$$\|c_h\|_{L^2(\Omega)}\le\|c_h\|^\theta_{H^1(\Omega)}\|c_h\|^{1-\theta}_{H^{-s}}\le M\|c_h\|^\theta_{H^1(\Omega)}\|c_h\|^{1-\theta}_{(C\cap W^{1,4}(\Omega))'},\qquad 0=\theta-s(1-\theta),$$

*or $\theta=s/(1+s)=(4+d)/(8+d)$. If $q<2(8+d)/(4+d)$ then $q\theta<2$, and Holder's inequality shows*

$$\|c_h\|_{L^q[0,T;L^2(\Omega)]}\le M\|c_h\|^\theta_{L^2[0,T;H^1(\Omega)]}\|c_h\|^{1-\theta}_{L^r[0,T;(C\cap W^{1,4}(\Omega))']},\quad r=2q(1-\theta)/(2-q\theta).$$

*It follows that $\{c_h\}_{h>0}$ is precompact in $L^q[0,T;L^2(\Omega)]$ for $q<22/7$ in three dimensions and $q<10/3$ in two dimensions.*

**3.3.1. Proof of Theorem 3.6.** We begin with the following lemma which establishes the crucial equicontinuity property required for compactness.

LEMMA 3.9 (Equicontinuity). *Let $H$ be a Hilbert space with inner-product $(\cdot,\cdot)_H$, $W$ be a Banach space, and $W \hookrightarrow H \hookrightarrow W'$ be dense embeddings. Let $0 = t^0 < t^1 < \ldots < t^N = T$ be a uniform partition of $[0,T]$, $W_h \subset W$ be a subspace, and $\ell \geq 0$. Fix $1 \leq p, q < \infty$ with $1/p + 1/q \geq 1$ and assume that $w_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1}, t^n; W_h]$ and*

$$\int_{t^{n-1}}^{t^n} (w_{ht}, v_h)_H + (w_{h+}^{n-1} - w_{h-}^{n-1}, v_{h+}^{n-1})_H = \int_{t^{n-1}}^{t^n} F_h(v_h),$$

*for all $v_h \in \mathcal{P}_\ell(t^{n-1}, t^n; W_h)$, where $F_h \in L^q[0,T; W_h']$.*

*Then for all $0 \leq \delta \leq T$ there exists a constant $M(\ell) > 0$ such that*

$$\sup_{v_h \in L^p[\delta,T;W_h]} \frac{\int_\delta^T (w_h(t) - w_h(t-\delta), v_h)_H \, dt}{\|v_h\|_{L^p[\delta,T;W]}} \leq M(\ell) \|F\|_{L^q[0,T;W_h']} \max(\frac{T}{N}, \delta)^{1/q'} \delta^{1/p'},$$

*where $p'$ and $q'$ are the dual exponents to $p$ and $q$ respectively.*

This is a variation of [19, Lemma 3.3] and the proof, which we omit, follows as in [19] with minor modification. Granted this technical lemma, we now prove Theorem 3.6.

*Proof of Theorem 3.6.* Since $w_h(t) \in W_h$, Lemma 3.9 and the bound upon $\|\Pi_h\|_{\mathcal{L}(W,W_h)}$ can be combined to show

$$\begin{aligned}
\Big( \int_\delta^T \|w_h(t) - w_h(t-\delta)\|_{W'}^{p'} \, dt \Big)^{1/p'} &= \sup_{v \in L^p[\delta,T;W]} \frac{\int_\delta^T (w_h(t) - w_h(t-\delta), v)_H \, dt}{\|v\|_{L^p[\delta,T;W]}} \\
&= \sup_{v \in L^p[\delta,T;W]} \frac{\int_\delta^T (w_h(t) - w_h(t-\delta), \Pi_h(v))_H \, dt}{\|\Pi_h(v)\|_{L^p[\delta,T;W]}} \times \frac{\|\Pi_h(v)\|_{L^p[\delta,T;W]}}{\|v\|_{L^p[\delta,T;W]}} \\
&\leq M(\ell) \|F\|_{L^q[0,T;W_h']} \max(\frac{T}{N}, \delta)^{1/q'} \delta^{1/p'}.
\end{aligned}$$

By assumption $p > 1$, so $p' < \infty$, and it follows that $\{w_h\}_{h>0}$ is equicontinuous in $L^{p'}[0,T;W']$ and bounded in $L^p[0,T;V]$, so compact in $L^{p'}[\theta, T-\theta, W']$ for any fixed $0 < \theta < T/2$, [19, Theorem 3.2].

Next, since $\{w_h\}_{h>0}$ is uniformly equicontinuous in the sense that

$$\int_\delta^T \|w_h(t) - w_h(t-\delta)\|_{W'}^{p'} \, dt \leq M \delta^\alpha,$$

with parameter $\alpha = 1$, it follows [19, Lemma 3.4] that $\{w_h\}_{h>0}$ is bounded in $L^r[0,T;W']$ for any $1 \leq r < \infty$, and hence is compact in $L^r[0,T;W']$ for all $1 \leq r < \infty$.

To establish compactness of $\{w_h\}_{h>0}$ in $L^p[0,T;H]$ recall [17] that $V \hookrightarrow\!\!\!\rightarrow H \hookrightarrow W'$ implies that for all $\epsilon > 0$ there exists $M(\epsilon) > 0$ such that

$$\|w_h(t)\|_H \leq \epsilon \|w_h(t)\|_V + M(\epsilon) \|w_h(t)\|_{W'},$$

so

$$\|w_h\|_{L^p[0,T;H]} \le \epsilon\|w_h\|_{L^p[0,T;V]} + M(\epsilon)\|w_h\|_{L^p[0,T;W']}.$$

Since $\{w_h\}_{h>0}$ is bounded in $L^p[0,T;V]$ and compact in $L^p[0,T;W']$ it follows that it is also compact in $L^p[0,T;H]$. $\square$

**3.4. Convergence of the Velocity and Pressure.** Theorem 3.7 allows passage to a subsequence for which the concentrations converge in $L^2[0,T;L^2(\Omega)]$ and, upon passing to a further subsequence if necessary, at almost every $(t,x) \in (0,T)\times\Omega$. The same is also true for the higher order schemes, and in this section we show that this is sufficient to establish (strong) convergence of the numerical approximations of the velocity and pressure.

THEOREM 3.10. *Let the data and coefficients satisfy Assumptions 1.1 and 1.2 respectively and suppose that the maximal time step $\Delta t$ tends to zero with the mesh parameter. Suppose that the sequence $\{c_h\}_{h>0} \subset L^2[0,T;L^2(\Omega)]$ converges pointwise almost everywhere to $c \in L^2[0,T;L^2(\Omega)]$, then the velocity and pressure computed using scheme (2.2)–(2.3) over a regular family of meshes converge strongly in $L^2[0,T;H(\Omega;\mathrm{div})]$ and $L^2[0,T;L^2(\Omega)]$ respectively.*

*Proof.* Let $\mathbb{U} = L^2[0,T;U]$ and $\mathbb{P} = L^2[0,T;L^2(\Omega)]$ and denote the finite element subspaces by

$$\mathbb{U}_h = \{\mathbf{u}_h \in \mathbb{U} \mid \mathbf{u}_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;U_h]\}, \qquad \text{and}$$
$$\mathbb{P}_h = \{p_h \in \mathbb{P} \mid p_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;P_h]\}.$$

Lemma 3.1 shows that the numerical approximations $\{(\mathbf{u}_h,\varphi_h)\}_{h>0}$ are bounded in $\mathbb{U}\times\mathbb{P}$, so we may pass to a subsequence for which $(\mathbf{u}_h,\varphi_h)$ converges weakly to a pair $(\mathbf{u},\varphi)$ in $\mathbb{U}\times\mathbb{P}$. Also, we remark that since $\mu(.)$ takes values in a compact set, it follows from the dominated convergence theorem that $\mu(c_h) \to \mu(c)$ in $L^r[0,T;L^r(\Omega)]$ for each $1 \le r < \infty$.

To show that the weak limits satisfy the limiting equation, fix $(\mathbf{v},\psi) \in C^\infty([0,T]\times\bar{\Omega}) \cap (\mathbb{U}\times\mathbb{P})$. Classical approximation theory shows that exists a sequence $((\mathbf{v}_h,\psi_h))_h \subset \mathbb{U}_h\times\mathbb{P}_h$ such that $(\mathbf{v}_h,\psi_h) \to (\mathbf{v},\psi)$ in $W^{1,\infty}((0,T)\times\Omega)$. In this situation we may pass to the limit term-by term in equations (2.2) and (2.3) to show that

$$\int_0^T ((\mathbf{K}^{-1}(c)\mathbf{u},\mathbf{v}) - (\varphi,\mathrm{div}(\mathbf{v}))) = -\int_0^T (\varphi_{\mathrm{D}},\mathbf{v}\cdot\mathbf{n})_{\Gamma_{\mathrm{D}}},$$
$$\int_0^T (\mathrm{div}(\mathbf{u}),\psi) = \int_0^T (f,\psi).$$

Since $C^\infty([0,T]\times\bar{\Omega}) \cap (\mathbb{U}\times\mathbb{P})$ is dense in $\mathbb{U}\times\mathbb{P}$, it follows that $(\mathbf{u},\varphi)$ is a weak solution of the mixed problem.

In order to establish strong convergence we introduce some notation to facilitate the use of abstract linear theory. For $c \in L^2[0,T;L^2(\Omega)]$ fixed, let $b(.,.;c) : (\mathbb{U}\times\mathbb{P})^2 \to \mathbb{R}$ be the bilinear form

$$b((\mathbf{u},\varphi),(\mathbf{v},\psi);c) = \int_0^T \big((\mathbf{K}^{-1}(c)\mathbf{u},\mathbf{v}) - (\varphi,\mathrm{div}(\mathbf{v})) + (\psi,\mathrm{div}(\mathbf{u}))\big).$$

Lemma 3.1 shows that $b(.,.;c)$ is coercive on $\mathbb{U}_h \times \mathbb{P}_h$ when $\mathbf{K}$ satisfies Assumption 1.2. Clearly, $b(\cdot,\cdot;c)$ is also continuous. Under these hypotheses, the Second Strang Lemma [6] states that

$$\|(\mathbf{u}-\mathbf{u}_h, \varphi-\varphi_h)\|_{\mathbb{U}\times\mathbb{P}} \le \inf_{(\mathbf{v}_h,\psi_h)\in\mathbb{U}_h\times\mathbb{P}_h} \|(\mathbf{u}-\mathbf{v}_h, \varphi-\psi_h)\|_{\mathbb{U}\times\mathbb{P}} + \sup_{(\mathbf{v}_h,\psi_h)\in\mathbb{U}_h\times\mathbb{P}_h} \frac{|b((\mathbf{u},\varphi),(\mathbf{v}_h,\psi_h);c) - b((\mathbf{u},\varphi),(\mathbf{v}_h,\psi_h);c_h)|}{\|(\mathbf{v}_h,\psi_h)\|_{\mathbb{U}\times\mathbb{P}}}.$$

The consistency error takes the form

$$b((\mathbf{u},\varphi),(\mathbf{v}_h,\psi_h);c) - b((\mathbf{u},\varphi),(\mathbf{v}_h,\psi_h);c_h) = \int_0^T \left((\mathbf{K}^{-1}(c)-\mathbf{K}^{-1}(c_h))\mathbf{u}, \mathbf{v}_h\right),$$

so

$$\|(\mathbf{u}-\mathbf{u}_h, \varphi-\varphi_h)\|_{\mathbb{U}\times\mathbb{P}} \le \inf_{(\mathbf{v}_h,\psi_h)\in\mathbb{U}_h\times\mathbb{P}_h} \|(\mathbf{u}-\mathbf{v}_h, \varphi-\psi_h)\|_{\mathbb{U}\times\mathbb{P}} + \|(\mathbf{K}^{-1}(c)-\mathbf{K}^{-1}(c_h))\mathbf{u}\|_{L^2[0,T;L^2(\Omega)]}.$$

The assumptions on $\mathbf{K}(.)$ guarantee that $|\mathbf{K}^{-1}(c_h)\mathbf{u}|^2$ converges pointwise to $|\mathbf{K}^{-1}(c)\mathbf{u}|^2$, and since $\mathbf{K}^{-1}(.)$ takes values in a compact set it follows that $|\mathbf{K}^{-1}(c_h)\mathbf{u}|^2 \le M|\mathbf{u}|^2$. Application of the dominated convergence theorem shows $\mathbf{K}^{-1}(c_h)\mathbf{u} \to \mathbf{K}^{-1}(c)\mathbf{u}$ in $L^2[0,T;L^2(\Omega)]$, and strong convergence of the velocity and pressure follows. □

**3.5. Convergence of the Concentration.** The compactness properties of the discrete solutions to the concentration were sufficient to prove strong convergence of the velocity and pressure. We now complete the "bootstrap" argument by showing that this, in turn, is sufficient to establish convergence of the discrete concentrations to a weak solution of equation (1.3).

THEOREM 3.11. *Let the data satisfy Assumptions 1.1, 1.2 and 2.1 and suppose that the maximal time step $\Delta t$ tends to zero with the mesh parameter. Then upon passage to a subsequence, the concentrations computed using scheme (2.2)–(2.4) over a regular family of meshes with $\ell = 0$ or $\ell = 1$ converge strongly in $L^2[0,T;L^2(\Omega)]$ and weakly in $L^2[0,T;H^1(\Omega)]$ to a weak solution $c$ of the concentration equation. In particular the triple $(\mathbf{u}_h, \varphi_h, c_h)$ converges to a weak solution of equations (1.1)-(1.3).*

*Proof.* Theorems 3.7, 3.10 and Lemma 3.5 allow passage to a subsequence for which the velocity and pressure converge strongly in $L^2[0,T;H(\Omega;\mathrm{div})]$ and $L^2[0,T;L^2(\Omega)]$ respectively to $\mathbf{u}$ and $\varphi$, and

$$\begin{aligned} c_h &\to c \text{ strongly in } L^2[0,T;L^2(\Omega)], \\ c_h &\rightharpoonup c \text{ weakly in } L^2[0,T;H^1(\Omega)], \\ D(\mathbf{u}_h)^{1/2}\nabla c_h &\rightharpoonup \boldsymbol{\eta} \text{ weakly in } L^2[0,T;L^2(\Omega)]. \end{aligned}$$

Since $D(\cdot)$ has linear growth and is continuous, the mapping $\mathbf{u} \mapsto D(\mathbf{u})$ is strongly continuous from $L^2[0,T;L^2(\Omega)]^d$ to $L^2[0,T;L^2(\Omega)]^{d\times d}$. It follows that the subsequence $(D(\mathbf{u}_h))_h$ converges to $D(\mathbf{u})$ strongly in $L^2[0,T;L^2(\Omega)]^{d\times d}$; in particular, $\boldsymbol{\eta} = D(\mathbf{u})^{1/2}\nabla c$.

Let $\mathbb{C}_h \subset L^2[0,T;C]$ denote the space of test functions for the numerical approximations of the concentration,

$$\mathbb{C}_h = \{d_h \in L^2[0,T;C] \mid d_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;C_h]\}.$$

If $d_h \in \mathbb{C}_h \cap \mathcal{C}[0,T;L^2(\Omega)]$ and $d_h(T) = 0$, then integrating the temporal term in (2.4) by parts and summing shows

$$\int_0^T \Big\{-(c_h,d_{ht})_H+(-c_h\mathbf{u}_h+D(\mathbf{u}_h)\nabla c_h,\nabla d_h)\Big\} = (c^0_{h-},d_h(0))_H+\int_0^T \Big\{(g,d_h)+(q_{\text{out}},d_h)_{\Gamma_{\text{out}}}\Big\}.$$

Let $d \in C^\infty[0,T;C^\infty(\bar\Omega)\cap C]$ and $d(T) = 0$. Classical approximation theory guarantees the existence of a sequence $\{d_h\}_h \subset \mathbb{C}_h \cap C^0[0,T;L^2(\Omega)]$ with $d_h(T) = 0$, that converges to $d$ in $W^{1,\infty}((0,T)\times\Omega)$ as $h$ tends to zero. We claim that we can then pass to the limit term by term in equation (3.5). Indeed, the right hand side is linear in $d_h$, and all but the third term on the left is a product of weakly and strongly converging terms. The third term also converges since it may be rewritten as

$$\int_0^T (D(\mathbf{u}_h)\nabla c_h,\nabla d_h) = \int_0^T (\nabla c_h, D(\mathbf{u}_h)\nabla d_h),$$

which is now a product of functions converging weakly and strongly in $L^2[0,T;L^2(\Omega)]$ respectively.

It follows that the limit $(c,\mathbf{u})$ satisfies

$$\int_0^T \Big\{-(c,d_t)+(-c\mathbf{u}+D(\mathbf{u})\nabla c,\nabla d)\Big\} = (c(0),d(0))_H+\int_0^T \Big\{(g,d)+(q_{\text{out}},d)_{\Gamma_{\text{out}}}\Big\},$$

for all smooth $d$ in $L^2[0,T;C]$ vanishing at $T$. As functions of $d$, all but the third term on the left are continuous for $d \in L^2[0,T;C]\cap H^1[0,T;C']$. If additionally $d \in L^4[0,T;W^{1,4}(\Omega)]$ then $D(\mathbf{u})^{1/2}\nabla d \in L^2[0,T;L^2(\Omega)]$, and the third term will be integrable since $D(\mathbf{u})^{1/2}\nabla c \in L^2[0,T;L^2(\Omega)]$. Since the smooth functions are dense in $L^2[0,T;C]\cap L^4[0,T;W^{1,4}(\Omega)]\cap H^1[0,T;C']$ it follows that the triple $(\mathbf{u},\varphi,c)$ is a weak solution of equations (1.1)-(1.3). □

**4. Higher Order Schemes.** We propose a higher order time stepping scheme that computes an approximation $(\mathbf{u}_h,\varphi_h)$ of the velocity and pressure using (2.2)-(2.3) and an approximation of the concentration $c_h$ satisfying the following equation:

$$\int_{t^{n-1}}^{t^n} (\phi c_{ht},d_h) + Q^n\left(-c_h\mathbf{u}_h+D(\mathbf{u}_h)\nabla c_h,\nabla d_h\right) + (c^{n-1}_{h+}-c^{n-1}_{h-},\phi d^{n-1}_{h+}) = \int_{t^{n-1}}^{t^n}\big((g,d_h)+(q_{\text{out}},d_h)_{\Gamma_{\text{out}}}\big). \tag{4.1}$$

This equation was obtained by modifying the low order scheme for the concentration by using a quadrature rule $Q^n$ to evaluate the nonlinear terms. The quadrature rule takes the general form

$$Q^n(f) = \Delta t^n \sum_{i=0}^{\ell} w_i f(t^{n-1} + \xi_i \Delta t^n),$$

where $\Delta t^n = t^{n-1} - t^n$, $\xi_i \in [0,1)$, and the weights $w_i$ are positive numbers. The quadrature rule, introduced below, has $\xi_0 = 0$ as a quadrature point, and will be exact on $\mathcal{P}_{2\ell}(0,1)$ so that the DG time stepping scheme will have formal order $\ell + 1$. The quadrature scheme is chosen to preserve the monotonicity of the elliptic term. Since this term is unbounded, non-monotone approximations of this term cannot be accommodated.

**4.1. Radau Quadrature.** To bound the jump terms in the DG time stepping scheme we need to select a test function $d_h$ satisfying $d_{h+}^{n-1} = c_{h+}^{n-1}$, and to facilitate this a Radau scheme with a quadrature point at the left hand end of the interval is utilized. For completeness, we now recall the Radau quadrature rule on the interval $[0,1]$.

Let $\{p_i\}_{i=0}^{\ell}$ be the polynomials on $(0,1)$ with $deg(p_i) = i$ which are orthonormal with respect to the inner product

$$(f,g) = \int_0^1 f(\xi) g(\xi)\, \xi \, d\xi.$$

Define the quadrature points $\{\xi_i\}_{i=0}^{\ell}$ to be the roots of $\xi p_\ell(\xi)$ and select the weights $w_i$ so that the quadrature rule

$$Q(f) = \sum_{i=0}^{\ell} w_i f(\xi_i),$$

is exact on $\mathcal{P}_\ell(0,1)$. This Radau scheme has positive weights and is exact on $\mathcal{P}_{2\ell}(0,1)$.

**Example:** If $\ell = 1$ then $\xi_0 = 0$, $\xi_1 = 2/3$ with weights $w_0 = 1/4$ and $w_1 = 3/4$. If $\ell = 2$ then

$$\begin{array}{lll} \xi_0 = 0, & \xi_1 = 4/9 + \sqrt{6}/36, & \xi_2 = 4/9 - \sqrt{6}/36, \\ w_0 = 1/9, & w_1 = 3/5 - \sqrt{6}/10, & w_2 = 3/5 + \sqrt{6}/10. \end{array}$$

□

Let $\Phi_i \in \mathcal{P}_\ell(0,1)$ denote the Lagrange interpolation functions satisfying $\Phi_i(\xi_j) = \delta_{ij}$, $0 \le i,j \le \ell$. Then the quadrature weights are $w_i = \int_0^1 \Phi_i$. Moreover, if $I : \mathcal{C}[0,1] \to \mathcal{P}_\ell(0,1)$ is the associated Lagrange interpolation operator;

$$I(f)(\xi) = \sum_{i=0}^{\ell} \Phi_i(\xi) f(\xi_i),$$

then it is easy to check that

$$I(f)(0) = f(0),$$
$$\int_0^1 I(f)p = Q(fp), \quad p \in \mathcal{P}_\ell(0,1),$$
$$\int_0^1 I(p) = Q(p), \quad p \in \mathcal{P}_{2\ell}(0,1),$$
$$Q(fg) \le Q(f^2)^{1/2} Q(g^2)^{1/2}.$$

Notation 4.1. *Given a partition $0 = t^0 < t^1 < \ldots < t^N = T$*

1. *$Q^n(.)$ will denote the quadrature scheme on $[t^{n-1}, t^n]$ obtained from the Radau quadrature $Q$ using the natural affine change of variables. That is, the scheme with weights $w_i \Delta t^n$ and quadrature points $s^i = t^{n-1} + \xi_i \Delta t^n$, where $\Delta t^n = t^n - t^{n-1}$. $Q_{\Delta t}(.)$ will denote the composite scheme on $[0,T]$, with $\Delta t = \max_n \Delta t^n$.*
2. *$I^n : \mathcal{C}[t^{n-1}, t^n] \to \mathcal{P}_\ell(t^{n-1}, t^n)$ will denote the corresponding Lagrange interpolation operator with interpolation points $s^i = t^{n-1} + \xi_i \Delta t^n$. The corresponding piecewise polynomial interpolant on $[0,T]$ is denoted by $I_{\Delta t}$.*

Properties of quadrature rule and associated interpolation operator, given above, are conserved by the change of variables. In particular, we have:

$$I^n(f)(t^{n-1}) = f(t^{n-1}_+), \tag{4.2}$$
$$\int_{t^{n-1}}^{t^n} I^n(f)p = Q^n(fp), \quad \forall p \in \mathcal{P}_\ell(t^{n-1}, t^n), \tag{4.3}$$
$$\int_{t^{n-1}}^{t^n} I^n(p) = \int_{t^{n-1}}^{t^n} p, \quad \forall p \in \mathcal{P}_{2\ell}(t^{n-1}, t^n), \tag{4.4}$$
$$Q^n(fg) \le (Q^n(f^2))^{1/2} (Q^n(g^2))^{1/2}. \tag{4.5}$$

We will also use the following property, for any spatial norm $\| \cdot \| = (.,.)^{1/2}$ on a Hilbert space:

$$\int_{t^{n-1}}^{t^n} \|I^n(f)\|^2 = Q^n(\|f\|^2). \tag{4.6}$$

**4.2. Stability of Schemes with Quadrature.** The quadrature rules were constructed so that the discrete scheme would be stable. The following analog of Lemma 3.5 requires the right hand side $f$ of the Darcy equation to be bounded in $L^\infty[0,T; L^\infty(\Omega)]$ instead of $L^1[0,T; L^\infty(\Omega)]$. This additional hypothesis is required since, unlike the low order schemes, bounding $c_h$ at the partition points $t^n$ is not sufficient to bound $c_h$ in $L^\infty[0,T; C_h]$.

Lemma 4.2 (Stability of high order schemes). *Let the data and coefficients satisfy Assumptions 1.1 and 1.2 respectively and assume additionally that $f \in L^\infty[0,T; L^\infty(\Omega)]$.*

*Then there exist positive constants $M_1, M_2$, independent of $h$ and $\Delta t$, such that the concentrations computed using equation (4.1), satisfy*

$$\max_{1\le n\le N} \|\phi^{1/2} c_{h-}^n\|_{L^2(\Omega)}^2 + \sum_{i=0}^{N-1} \|[\phi^{1/2} c_h^n]\|_{L^2(\Omega)}^2 + \int_0^T \left(\|c_h\|_{L^2(\Omega)}^2 + \|I_{\Delta t}(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|_{L^2(\Omega)}^2\right)$$
$$\le M_1 \exp((1+M_2\|f\|_{L^\infty[0,T;L^\infty(\Omega)]})T)\left(\|\phi^{1/2} c_{h-}^0\|_{L^2(\Omega)} + \|g\|_{L^2[0,T;C']} + \|q_{\text{out}}\|_{L^2[0,T;H^{-1/2}(\Gamma_{\text{out}})]}\right),$$

*provided* $(1 + M_2\|f\|_{L^\infty[0,T;L^\infty(\Omega)]})\Delta t < 1$. *In particular,*

$$\max_{0\le n\le N} \|c_{h-}^n\|_{L^2(\Omega)}, \qquad \|c_h\|_{L^2[0,T;H^1(\Omega)]}, \qquad \textit{and} \qquad \|I_{\Delta t}(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|_{L^2[0,T;L^2(\Omega)]}$$

*are bounded independently of $h$ and $\Delta t$.*

*Proof.* The idea is to select the test function to be an approximation of $\exp(-\lambda t)c_h(t)$ on $[t^{n-1}, t^n)$ where $\lambda > 0$ is to be specified. Specifically, fix $\lambda > 0$, define $\omega(t) = 1-\lambda(t-t^{n-1})$ and select the test function for the concentration equation (4.1) to be $d_h = I_{\Delta t}(\omega c_h)$. Since $\omega c_h \in \mathcal{P}_{\ell+1}[t^{n-1}, t^n; C_h]$ and $c_{ht} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; C_h]$, from (4.3) and (4.4), we have

$$\begin{aligned}\int_{t^{n-1}}^{t^n} (\phi c_{ht}, d_h) &= \int_{t^{n-1}}^{t^n} (\phi c_{ht}, I^n(\omega c_h)) = Q^n((\phi c_{ht}, \omega c_h)) = \int_{t^{n-1}}^{t^n} (\phi c_{ht}, \omega c_h) \\ &= \frac{1}{2}(1-\lambda\Delta t^n)\|\phi^{1/2} c_{h+}^n\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\phi^{1/2} c_{h+}^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_{t^{n-1}}^{t^n} \lambda\|\phi^{1/2} c_h^2\|_{L^2(\Omega)}.\end{aligned}$$

In addition, from (4.2) we find

$$(c_{h+}^{n-1} - c_{h-}^{n-1}, \phi d_{h+}^{n-1}) = (c_{h+}^{n-1} - c_{h-}^{n-1}, \phi c_{h+}^{n-1}) = \frac{1}{2}\|\phi^{1/2} c_{h+}^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\phi^{1/2}[c_h^{n-1}]\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\phi^{1/2} c_{h-}^{n-1}\|_{L^2(\Omega)}^2.$$

Combining the above shows

$$\begin{aligned}\int_{t^{n-1}}^{t^n} (\phi c_{ht}, d_h) + (c_{h+}^{n-1} - c_{h-}^{n-1}, \phi d_{h+}^{n-1}) &= (1/2)(1-\lambda\Delta t^n)\|\phi^{1/2} c_{h-}^n\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (\lambda/2)\|\phi^{1/2} c_h\|_{L^2(\Omega)}^2 \\ &\quad + (1/2)\|\phi^{1/2}[c_h^{n-1}]\|_{L^2(\Omega)}^2 - (1/2)\|\phi^{1/2} c_{h-}^{n-1}\|_{L^2(\Omega)}^2.\end{aligned}$$

We next show that the quadrature rule preserves the monotonicity of the principle term provided the time steps are sufficiently small. We bound the convective term using the estimates (3.1) and (3.2) and property (4.4) of the quadrature rule. Recall that $\Pi_h : L^2[t^{n-1}, t^n; L^2(\Omega)] \to \mathcal{P}_\ell[t^{n-1}, t^n; P_h]$ is the orthogonal projection.

$$\begin{aligned}Q^n((c_h\mathbf{u}_h, \nabla d_h)) &= -\frac{1}{2}Q^n((\operatorname{div}(\mathbf{u}_h), \omega c_h^2)) = -\frac{1}{2}Q^n((\Pi_h(f), \omega c_h^2)) \\ &\le (1/2)\|\Pi_h(f)\|_{L^\infty[0,T;L^\infty(\Omega)]} Q^n(\|c_h\|_{L^2(\Omega)}^2) \\ &\le (M_1/2)\|f\|_{L^\infty[0,T;L^\infty(\Omega)]}\|c_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}^2.\end{aligned}$$

The diffusive term is bounded using property (4.6) of the quadrature rule.

$$\begin{aligned}Q^n((D(\mathbf{u}_h)\nabla c_h, \nabla d_h)) &= \sum_{i=0}^{\ell} w_i \Delta t^n (1-\lambda(s_i - t^{n-1}))\|D(\mathbf{u}_h)^{1/2}\nabla c_h\|^2_{L^2(\Omega)}|_{t=s^i} \\ &\geq (1-\lambda\Delta t^n)\sum_{i=0}^{\ell} w_i \Delta t^n \|D(\mathbf{u}_h)^{1/2}\nabla c_h|_{t=s^i}\|^2_{L^2(\Omega)} \\ &= (1-\lambda\Delta t^n)\int_{t^{n-1}}^{t^n} \|I^n(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|^2_{L^2(\Omega)}.\end{aligned}$$

In the above it is assumed that $\lambda\Delta t^n \leq 1$. We now combine the inequalities above and obtain, for $\lambda\Delta t^n \leq 1/2$,

$$\begin{aligned}&(1-\lambda\Delta t^n)\|\phi^{1/2}c^n_{h-}\|^2_{L^2(\Omega)} + \|\phi^{1/2}[c_h^{n-1}]\|^2_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n}\Big(\lambda\|\phi^{1/2}c_h\|^2_{L^2(\Omega)} + \|I^n(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|^2_{L^2(\Omega)}\Big) \\ &\leq \|\phi^{1/2}c^{n-1}_{h-}\|^2_{L^2(\Omega)} \qquad (4.7) \\ &\quad + \int_{t^{n-1}}^{t^n}\left\{M\|f\|_{L^\infty[0,T;L^\infty(\Omega)]}\|c_h\|^2_{L^2(\Omega)} + \frac{2}{d_m}(\|g\|_{C'} + \|q_{\text{out}}\|_{H^{-1/2}(\Gamma_{\text{out}})})^2 + \frac{d_m}{2}\|c_h\|^2_{H^1(\Omega)}\right\}.\end{aligned}$$

Pick $\lambda = 1 + (\frac{d_m}{2} + M\|f\|_{L^\infty[0,T;L^\infty(\Omega)]})/\phi_0$ and note that

$$\begin{aligned}\int_{t^{n-1}}^{t^n}\|I^n(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|^2_{L^2(\Omega)} &= Q^n\left(\|D(\mathbf{u}_h)^{1/2}\nabla c_h\|^2_{L^2(\Omega)}\right) \\ &\geq d_m Q^n\left(\|\nabla c_h\|^2_{L^2(\Omega)}\right) = d_m\int_{t^{n-1}}^{t^n}\|\nabla c_h\|^2_{L^2(\Omega)},\end{aligned}$$

so that we can write

$$\begin{aligned}\int_{t^{n-1}}^{t^n}\big(\lambda\|\phi^{1/2}c_h\|^2_{L^2(\Omega)} + \|I^n(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|^2_{L^2(\Omega)}\big) &\geq (1+M\|f\|_{L^\infty[0,T;L^\infty(\Omega)]})\int_{t^{n-1}}^{t^n}\|\phi^{1/2}\nabla c_h\|^2_{L^2(\Omega)} \\ &\quad + \frac{d_m}{2}\int_{t^{n-1}}^{t^n}\|c_h\|^2_{H^1(\Omega)} + \frac{1}{2}\int_{t^{n-1}}^{t^n}\|I^n(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore (4.7) becomes:

$$\begin{aligned}&(1-\lambda\Delta t^n)\|\phi^{1/2}c^n_{h-}\|^2_{L^2(\Omega)} + \|\phi^{1/2}[c_h^{n-1}]\|^2_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n}\big(\|\phi^{1/2}c_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|I^n(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|^2_{L^2(\Omega)}\big) \\ &\qquad \leq \|\phi^{1/2}c^{n-1}_{h-}\|^2_{L^2(\Omega)} + \frac{2}{d_m}\int_{t^{n-1}}^{t^n}(\|g\|_{C'} + \|q_{\text{out}}\|_{H^{-1/2}(\Gamma_{\text{out}})})^2.\end{aligned}$$

The discrete Gronwall inequality, Lemma 3.4, completes the proof. □

**4.3. Compactness of the Concentration.** Theorem 3.6 will be used to show compactness of the concentrations. The stability estimates of Lemma 4.2 provide the bounds required upon $\{c_h\}_{h>0}$, and, as for the low order time stepping schemes, these will be sufficient to bound the spatial terms in equation (4.1) to establish the third hypothesis of Theorem 3.6.

THEOREM 4.3. *Let the data and coefficients satisfy Assumptions 1.1 and 1.2 respectively and suppose additionally that* $f \in L^\infty[0,T;L^\infty(\Omega)]$ *and that the maximal time step* $\Delta t$ *tends to zero with the mesh parameter. Then the concentrations* $\{c_h\}_{h>0}$ *computed using (2.2), (2.3) and (4.1) are precompact in* $L^2[0,T;L^2(\Omega)] \cap L^r[0,T;(C\cap W^{1,4}(\Omega))']$ *for any* $1 \le r < \infty$.

*Proof.* The proof of Theorem 3.7 will carry over provided the terms in equation (4.1) computed using the quadrature rule can be bounded so that the third hypothesis of Theorem 3.6 holds. We define a function $\hat{F}_h$ by

$$\hat{F}_h(d_h) = I_{\Delta t}(-c_h \mathbf{u}_h + D(\mathbf{u}_h)\nabla c_h, \nabla d_h), \tag{4.8}$$

and we have

$$\int_0^T \hat{F}_h(d_h) = Q_{\Delta t}((-c_h\mathbf{u}_h + D(\mathbf{u}_h)\nabla c_h, \nabla d_h)).$$

It suffices to show that $\hat{F}_h \in L^1(0,T;C_h')$. Fixing $d_h \in \{L^\infty[0,T;W^{1,4}(\Omega)] \mid d_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;C_h]\}$, the convection term is estimated by

$$\begin{aligned}
Q_{\Delta t}((-c_h\mathbf{u}_h,\nabla d_h)) &\le Q_{\Delta t}(\|c_h\|_{L^4(\Omega)}\|\mathbf{u}_h\|_{L^2(\Omega)}\|\nabla d_h\|_{L^4(\Omega)}) \\
&\le MQ_{\Delta t}(\|c_h\|_{H^1(\Omega)}\|\mathbf{u}_h\|_{L^2(\Omega)})\|\nabla d_h\|_{L^\infty[0,T;L^4(\Omega)]} \\
&\le M(Q_{\Delta t}(\|c_h\|^2_{H^1(\Omega)}))^{1/2}(Q_{\Delta t}(\|\mathbf{u}_h\|^2_{L^2(\Omega)}))^{1/2}\|\nabla d_h\|_{L^\infty[0,T;L^4(\Omega)]} \\
&\le M\|c_h\|_{L^2[0,T;H^1(\Omega)]}\|\mathbf{u}_h\|_{L^2[0,T;L^2(\Omega)]}\|\nabla d_h\|_{L^\infty[0,T;L^4(\Omega)]},
\end{aligned}$$

where the last line follows since $\|c_h\|^2_{H^1(\Omega)}$ and $\|\mathbf{u}_h\|^2_{L^2(\Omega)}$ are piecewise polynomials of degree $2\ell$ so are integrated exactly by the quadrature rule.

To estimate the second term of $\hat{F}_h$, use properties (4.5) and (4.6) of the quadrature scheme to write

$$\begin{aligned}
Q_{\Delta t}\left((D(\mathbf{u}_h)\nabla c_h,\nabla d_h)\right) &= Q_{\Delta t}\left((D(\mathbf{u}_h)^{1/2}\nabla c_h, D(\mathbf{u}_h)^{1/2}\nabla d_h)\right) \\
&\le (Q_{\Delta t}(\|D(\mathbf{u}_h)^{1/2}\nabla c_h\|^2_{L^2(\Omega)}))^{1/2}(Q_{\Delta t}(\|D(\mathbf{u}_h)^{1/2}\nabla d_h\|^2_{L^2(\Omega)}))^{1/2} \\
&\le \|I_{\Delta t}(D(\mathbf{u}_h)^{1/2}\nabla c_h)\|_{L^2[0,T;L^2(\Omega)]}(Q_{\Delta t}(\|D(\mathbf{u}_h)^{1/2}\nabla d_h\|^2_{L^2(\Omega)}))^{1/2}.
\end{aligned}$$

The stability estimate bounds the first term. To estimate the second term use (1.6) to obtain

$$\begin{aligned}
Q_{\Delta t}(\|D(\mathbf{u}_h)^{1/2}\nabla d_h\|^2_{L^2(\Omega)}) &\le MQ_{\Delta t}\left(\|\nabla d_h\|^2_{L^2(\Omega)} + \|\mathbf{u}_h\|_{L^2(\Omega)}\|\nabla d_h\|^2_{L^4(\Omega)}\right) \\
&\le M\left(\|\nabla d_h\|^2_{L^2[0,T;L^2(\Omega)]} + \|\mathbf{u}_h\|_{L^2[0,T;L^2(\Omega)]}\|\nabla d_h\|^2_{L^\infty[0,T;L^4(\Omega)]}\right).
\end{aligned}$$

In the last line we use the fact that $\|\nabla d_h\|^2_{L^2(\Omega)}$ and $\|\mathbf{u}_h\|^2_{L^2(\Omega)}$ are piecewise polynomials of degree $2\ell$, so are integrated exactly by the quadrature rule. The rest of the proof is the same as in the proof of Theorem 3.7. ▯

**4.4. Convergence of the Concentration.** Compactness of the concentrations allows passage to a subsequence for which $c_h \to c$ in $L^2[0,T;L^2(\Omega)]$ and converges pointwise at almost every $(t,x) \in (0,T)\times\Omega$. Theorem 3.10 then shows that the corresponding velocities and pressures $(\mathbf{u}_h, \varphi_h)$ then converge strongly in $L^2(\Omega)[0,T;H(\Omega;\mathrm{div})]\times L^2[0,T;L^2(\Omega)]$ to a weak solution of equations (1.1) and (1.2). The following theorem shows that the concentrations will then converge to a weak solution of equation (1.3).

THEOREM 4.4. *Let the data satisfy Assumptions 1.1, 1.2 and 2.1 and suppose additionally that $f \in L^\infty[0,T;L^\infty(\Omega)]$ and that the maximal time step $\Delta t$ tends to zero with the mesh parameter. Then upon passage to a subsequence, the concentrations computed using scheme (2.2), (2.3) and (4.1) over a regular family of meshes converge strongly in $L^2[0,T;L^2(\Omega)]$ and weakly in $L^2[0,T;H^1(\Omega)]$ to a weak solution $c$ of the concentration equation. In particular the triple $(\mathbf{u}_h, \varphi_h, c_h)$ converges to a weak solution of equations (1.1)-(1.3).*

*Proof.* The argument is the similar to the one used to prove convergence of the low order schemes in Theorem 3.11. The major difference is that in the present situation some terms are evaluated using quadrature, so it suffices to show that these converge to the correct limits.

Using the stability and compactness properties estimates we may pass to a subsequence for which

$$\begin{aligned}
\mathbf{u}_h &\to \mathbf{u} \text{ strongly in } L^2[0,T;L^2(\Omega)]\\
c_h &\to c \text{ strongly in } L^2[0,T;L^2(\Omega)]\\
c_h &\rightharpoonup c \text{ weakly in } L^2[0,T;H^1(\Omega)]\\
I_{\Delta t}(D(\mathbf{u}_h)^{1/2}\nabla c_h) &\rightharpoonup \boldsymbol{\eta} \text{ weakly in } L^2[0,T;L^2(\Omega)].
\end{aligned}$$

If $d_h \in \mathbb{C}_h \cap \mathcal{C}[0,T;L^2(\Omega)]$ converges to $d$ in $W^{1,\infty}((0,T)\times\Omega)$, we need to show that

$$\lim_{h,\Delta t\to 0} \int_0^T \hat{F}_h(d_h) = \int_0^T (-c\mathbf{u} + D(\mathbf{u})\nabla c, \nabla d)\,,$$

where $\hat{F}_h$ is the function specified in equation (4.8). Let $\bar{d}_h$ be piecewise constant in time on each interval $(t^{n-1}, t^n)$ and take the average value of $d_h$. Then $\bar{d}_h \to d$ strongly in $L^\infty[0,T;W^{1,\infty}(\Omega)]$, and in the proof of Theorem 4.3 it was shown that

$$\left|\int_0^T \hat{F}_h(d_h - \bar{d}_h)\right| \le M\|d_h - \bar{d}_h\|_{L^\infty[0,T;W^{1,4}(\Omega)]},$$

which converges to zero as $h$ tends to zero. It then suffices to show

$$\lim_{h,\Delta t\to 0} \int_0^T \hat{F}_h(\bar{d}_h) = \int_0^T (-c\mathbf{u} + D(\mathbf{u})\nabla c, \nabla d)\,.$$

On each interval, $(c_h \mathbf{u}_h, \nabla \bar{d}_h) \in \mathcal{P}_{2\ell}(t^{n-1}, t^n)$, so the quadrature rule is exact, and

$$Q_{\Delta t}\left((-c_h \mathbf{u}_h, \nabla \bar{d}_h)\right) = \int_0^T \left(-c_h \mathbf{u}_h, \nabla \bar{d}_h\right) \to \int_0^T \left(-c\mathbf{u}, \nabla d\right).$$

To establish convergence of the principle term, let $\bar{\mathbf{u}}_h$ be piecewise constant in time on each interval $(t^{n-1}, t^n)$ and take the time average value of $\mathbf{u}_h$. Then $\{\bar{\mathbf{u}}_h\}_h$ converges weakly to $\mathbf{u}$ in $L^2[0,T; L^2(\Omega)]$, and a calculation shows

$$\|\bar{\mathbf{u}}_h\|_{L^2[0,T;L^2(\Omega)]} \le \|\mathbf{u}_h\|_{L^2[0,T;L^2(\Omega)]} \to \|\mathbf{u}\|_{L^2[0,T;L^2(\Omega)]};$$

so $\bar{\mathbf{u}}_h \to \mathbf{u}$ strongly in $L^2[0,T; L^2(\Omega)]$. Then write the principle term as

$$Q_{\Delta t}\left((D(\mathbf{u}_h)\nabla c_h, \nabla \bar{d}_h)\right) = Q_{\Delta t}\left(((D(\mathbf{u}_h) - D(\bar{\mathbf{u}}_h))\nabla c_h, \nabla \bar{d}_h)\right) + \int_0^T (D(\bar{\mathbf{u}}_h)\nabla c_h, \nabla \bar{d}_h), \tag{4.9}$$

where we use the fact that on each interval $(D(\bar{\mathbf{u}}_h)\nabla c_h, \nabla \bar{d}_h) \in \mathcal{P}_\ell(t^{n-1}, t^n)$ so is integrated exactly by the quadrature rule. Since $D : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfies the Caratheodory conditions with linear growth, it follows that $D(\bar{\mathbf{u}}_h) \to D(\mathbf{u})$ strongly in $L^2[0,T; L^2(\Omega)]$, so

$$\int_0^T (D(\bar{\mathbf{u}}_h)\nabla c_h, \nabla \bar{d}_h) \to \int_0^T (D(\mathbf{u})\nabla c, \nabla d).$$

It then suffices to show that the first term on the right of equation (4.9) vanishes in the limit. To do this we use the Lipschitz continuity of $D(\cdot)$ established in Lemma 1.3). We compute

$$\begin{aligned}
&Q_{\Delta t}\big(((D(\mathbf{u}_h) - D(\bar{\mathbf{u}}_h))\nabla c_h, \nabla \bar{d}_h)\big) \\
&\qquad \le M Q_{\Delta t}\left(\|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^2(\Omega)} \|\nabla c_h\|_{L^2(\Omega)}\right) \|\nabla \bar{d}_h\|_{L^\infty[0,T;L^\infty(\Omega)]} \\
&\qquad \le M Q_{\Delta t}\left(\|\mathbf{u}_h - \bar{\mathbf{u}}_h\|^2_{L^2(\Omega)}\right)^{1/2} Q_{\Delta t}\left(\|\nabla c_h\|^2_{L^2(\Omega)}\right)^{1/2} \|\nabla \bar{d}_h\|_{L^\infty[0,T;L^\infty(\Omega)]} \\
&\qquad \le M \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^2[0,T;L^2(\Omega)]} \|\nabla c_h\|_{L^2[0,T;L^2(\Omega)]} \|\nabla \bar{d}_h\|_{L^\infty[0,T;L^\infty(\Omega)]},
\end{aligned}$$

The second line uses the Cauchy Schwarz inequality and the last line follows since $\|\mathbf{u}_h - \bar{\mathbf{u}}_h\|^2_{L^2(\Omega)}$ and $\|\nabla c_h\|^2_{L^2(\Omega)}$ are in $\mathcal{P}_{2\ell}(t^{n-1}, t^n)$ on each interval. It follows that this term vanishes in the limit, so, upon passing to a subsequence, the high order numerical schemes with quadrature converge strongly in $L^2[0,T; H(\Omega; \mathrm{div})] \times L^2[0,T; L^2(\Omega)] \times L^2[0,T; L^2(\Omega)]$. To conclude that the limit $(\mathbf{u}, \varphi, c)$ is a weak solution of equations (1.1)-(1.3) as defined in Section 1.3 we need to verify that $D^{1/2}(\mathbf{u})\nabla c \in L^2[0,T; L^2(\Omega)]$. It suffices to show that $\boldsymbol{\eta} = D^{1/2}(\mathbf{u})\nabla c$.

Let $\mathbf{w} \in L^2[0,T; L^2(\Omega)]$ be continuous, and, as above, let $\bar{\mathbf{w}}_h$ be piecewise constant on the intervals $(t^{n-1}, t^n)$ and take the average value of $\bar{\mathbf{w}}$. Then following the line of argument in the previous paragraph,

$$\begin{aligned}\int_0^T (\boldsymbol{\eta}, \mathbf{w}) &= \lim_{h\to 0} \int_0^T \left( I_{\Delta t}(D^{1/2}(\mathbf{u}_h)\nabla c_h), \bar{\mathbf{w}}_h \right) \\ &= \lim_{h\to 0} \int_0^T \left( D^{1/2}(\bar{\mathbf{u}}_h)\nabla c_h, \bar{\mathbf{w}}_h \right) + \lim_{h\to 0} Q_{\Delta t} \left( ((D(\mathbf{u}_h)^{1/2} - D(\bar{\mathbf{u}}_h)^{1/2})\nabla c_h, \bar{\mathbf{w}}) \right) \\ &= \int_0^T \left( D^{1/2}(\mathbf{u})\nabla c, \mathbf{w} \right) + \lim_{h\to 0} Q_{\Delta t} \left( ((D(\mathbf{u}_h)^{1/2} - D(\bar{\mathbf{u}}_h)^{1/2})\nabla c_h, \bar{\mathbf{w}}) \right).\end{aligned}$$

Lemma 1.3 shows that $D^{1/2}(\cdot)$ is Lipschitz, so

$$\begin{aligned}&|Q_{\Delta t}\left( ((D(\mathbf{u}_h)^{1/2} - D(\bar{\mathbf{u}}_h)^{1/2})\nabla c_h, \bar{\mathbf{w}}) \right)| \\ &\qquad \leq Q_{\Delta t} \left( \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^2(\Omega)} \|\nabla c_h\|_{L^2(\Omega)} \right) \|\bar{\mathbf{w}}_h\|_{L^\infty[0,T;L^\infty(\Omega)]} \\ &\qquad \leq Q_{\Delta t} \left( \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|^2_{L^2(\Omega)} \right)^{1/2} Q_{\Delta t} \left( \|\nabla c_h\|^2_{L^2(\Omega)} \right)^{1/2} \|\bar{\mathbf{w}}_h\|_{L^\infty[0,T;L^\infty(\Omega)]} \\ &\qquad \leq \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_{L^2[0,T;L^2(\Omega)]} \|\nabla c_h\|_{L^2[0,T;L^2(\Omega)]} \|\bar{\mathbf{w}}_h\|_{L^\infty[0,T;L^\infty(\Omega)]},\end{aligned}$$

which converges to zero as $h$ tends to zero. This shows

$$\int_0^T (\boldsymbol{\eta}, \mathbf{w}) = \int_0^T \left( D^{1/2}(\mathbf{u})\nabla c, \mathbf{w} \right),$$

for all smooth $\mathbf{w} \in L^2[0,T;L^2(\Omega)]$ from which it follows that $D^{1/2}(\mathbf{u})\nabla c = \boldsymbol{\eta} \in L^2[0,T;L^2(\Omega)]$.
□

**5. Conclusions.** This paper formulates and analyzes a numerical method for solving the miscible displacement problem. The proposed discretization employs a discontinuous Galerkin method in time and a combined mixed method and finite element method in space. Stability and convergence of the numerical approximation of pressure, velocity and concentration are obtained under minimal regularity.

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