Test 3
August 11

Name:

1. Solve each of the following differential equations

(a)

$$y' + y = x^{3/2}e^{-x} \quad y(0) = -2$$

Linear diff. eq. +2

$I(x) = e^{\int 1 dx} = e^x$ +2

$e^x y' + e^x y = x^{3/2}$

$\int (e^x y)' = \int x^{3/2}$

$e^x y = \frac{2}{5} x^{5/2} + C$ +2

$y = \frac{2}{5} x^{5/2} e^{-x} + Ce^{-x}$ +1

$y(0) = -2$:

$-2 = 0 + C$

$C = -2$ +2

$y = \frac{2}{5} x^{5/2} e^{-x} - 2e^{-x}$ +1

(b)

$$y' = \frac{x^2+1}{1+\sin y}$$

Separable +3

$$\int 1+\sin y \, dy = \int x^2+1 \, dx \quad +4$$

$$y - \cos y = \frac{x^3}{3} + x + C \quad +3$$

(c)

$$y'' - y' - 6y = 0$$

Aux. Eqn: $r^2 - r - 6 = 0$ + 3

$(r-3)(r+2) = 0$

$r = +3$ or -2 + 3

$y(x) = c_1 e^{3x} + c_2 e^{-2x}$ + 4

(d)

$$y'' - 4y' + 5y = 0$$

Aux. Eqn. $r^2 - 4r + 5 = 0$ +3

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 5}}{2} = 2 \pm \frac{\sqrt{-4}}{2} = 2 \pm i$$ +2

$$y = e^{2x}(c_1 \cos x + c_2 \sin x)$$ +5

(e)

$$y'' - 4y' + 4y = \sin(4x)$$

Find y_c :

Aux eqn: $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$

$y_c = c_1 e^{2x} + c_2 x e^{2x} \quad + 3$

Find y_p :

Try $y_p = A\cos 4x + B\sin 4x \quad + 2$

$y_p' = -4A\sin 4x + 4B\cos 4x$

$y_p'' = -16A\cos 4x - 16B\sin 4x$

$(-16A\cos 4x - 16B\sin 4x) - 4(-4A\sin 4x + 4B\cos 4x) + 4(A\cos 4x + B\sin 4x) = \sin 4x$
$+2$

$-16B + 16A + 4B = 1 \rightarrow 16A - 12B = 1$

$-16A - 16B + 4A = 0 \rightarrow -12A - 16B = 0 \rightarrow A = -\frac{4}{3}B$

$16\left(-\frac{4}{3}B\right) - 12B = 1$

$-\frac{100}{3}B = 1 \rightarrow B = \frac{-3}{100}$

$A = -\frac{4}{3} \cdot \frac{-3}{100} = \frac{1}{25}$

$y_p = \frac{\cos 4x}{25} - \frac{3\sin x}{100} + 2$

$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \frac{\cos 4x}{25} - \frac{3\sin x}{100} + 1$

2. A tank initially contains 100L of water, with 10 kg of salt. Brine with a concentration of 0.1 kg/L enters the tank at a rate of 10 L/min. The tank drains at a rate of 20 L/min. Find a function describing how much salt is in the tank as a function of time. What is the domain of your function?

$y(t)$ = salt in kg after t minutes

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$= 0.1 \cdot 10 - \left(\frac{y}{100-10t}\right) 20$$ +2

$$y' + \left(\frac{20}{100-10t}\right) y = 1$$

$$I(t) = e^{\int \frac{20}{100-10t} dt} = e^{2\ln|100-10t|} = (100-10t)^{-2}$$ +2

$$\int \left((100-10t)^{-2} y\right)' = \int (100-10t)^{-2}$$

$$(100-10t)^{-2} y = \frac{-(100-10t)^{-1}}{10} + C$$ +2

$$y = \frac{-(100-10t)}{10} + C(100-10t)^{2}$$

$$= t - 10 + C(100-10t)^2$$

$y(0) = 10$: $10 = 0 - 10 + C(100-0)^2$

$$20 = C(100)^2$$ +2

$$C = \frac{20}{100^2} = \frac{2}{1000}$$

$$y = t - 10 + \frac{2(100-10t)^2}{1000}$$ +1

Domain is $0 \le t \le 10$ +1

3. Suppose pure sodium (Na) is pumped into a tank at a constant rate 5 kg/min. As it is pumped in, it reacts with chlorine (Cl) and it is converted into salt (NaCl) at a rate proportional to the concentration at the time (assume there is an infinite supply of Cl), with a proportionality constant of 2. Let $y(t)$ = the amount of pure sodium in the tank at time t

   (a) Explain why y is a solution to the following differential equation:

   $$y' = 5 - 2y$$

   (b) Solve this differential equation, and explain what happens as time goes on.

a) $y' =$ rate in − rate out   + 2

$= 5 \quad -2y$

b) $y' + 2y = 5$

$I(t) = e^{\int 2dt} = e^{2t}$

$(e^{2t} y)' = \int 5e^{2t} = \frac{5}{2}e^{2t} + C$ + 3

$\boxed{y = \frac{5}{2} + Ce^{-2t}}$ + 3

$\lim_{t\to\infty} y(t) = \frac{5}{2}$, so the amount of salt approaches $\frac{5}{2}$ kg, (no matter how much is in the tank initially)   + 2

4. A spring has a weight with a mass of 5 attached to the end of it. If x is the distance the weight is away from equilibrium, the resisting force of the spring is $-5x$.

   (a) Give an equation desctibing the position of the mass as a function of time.

   (b) You intend to damp the vibrations with a viscous fluid with damping force $-cx'$. Descibe what consequences choosing different values of c will have.

a) $ma = f$

$5x'' = -5x$ +2

$5x'' + 5x = 0$

$x'' + x = 0$

Aux eqn: $r^2 + 1 = 0$

$r = \pm i$

$x = c_1 \cos t + c_2 \sin t$ +3

b) $5x'' + cx' + 5x = 0$

Aux: $5r^2 + cr + 5 = 0$

$$r = \frac{-c \pm \sqrt{c^2 - 100}}{10}$$ +2

$c > 10$: overdamped

$c = 10$: critical damping

$0 \leq c < 10$: underdamped

+3