Test 2
July 28

Name:

1. Determine whether each of the following series converges or diverges. If a series converges, evaluate the sum, when possible.

   (a)

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{3n}}$$

This is a geometric series $+2$ with $a = \frac{9}{8} + 2$, $r = \frac{3}{8} + 3$

$|r| < 1$, so it converges to $\frac{a}{1-r} + 2$

$$= \frac{9}{8} \cdot \frac{1}{1-\frac{3}{8}} = \frac{9}{8} \cdot \frac{8}{5} = \frac{9}{5}$$

(b)

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

$f(x) = xe^{-x^2}$ is ≥ 0, cont., and eventually decreasing +1

$$\left(f'(x) = -2x^2 xe^{-x^2} < 0\right)$$

so we may apply the integral test: +5

$$\int_1^{\infty} xe^{-x^2} \overset{+1}{=} \lim_{t\to\infty} \int_1^t xe^{-x^2}$$

$$= \lim_{t\to\infty} \left[-\frac{e^{-x^2}}{2}\right]_1^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(-e^{-t^2} + e^{-1}\right) \quad +2$$

$$= \frac{1}{2e}$$

The integral converges.

$\therefore \sum$ converges +1

(c)

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+1)!}$$

Ratio test: +4

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1}}{(2n+3)!} \frac{(2n+1)!}{(-2)^n}\right|$$

$$= \left|\frac{-2}{(2n+3)(2n+2)}\right| \xrightarrow[+1]{n\to\infty} 0$$

+2

$\Rightarrow$ series converges

+3

OR: AST +3

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{(2n+1)!} \quad +1$$

- $\dfrac{2^n}{(2n+1)!} > 0$
- $\dfrac{2^n}{(2n+1)!} > \dfrac{2^{n+1}}{(2n+3)!}$ since $(2n+3)(2n+2) > 2$ +3
- $\lim_{n\to\infty} \dfrac{2^n}{(2n+1)!} = 0$ by comparison with $\dfrac{2}{n}$ +3

  $\dfrac{2^n}{(2n+1)!} < \dfrac{2}{n}$ for $n \geq 1$

$\therefore$ converges

(d)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^2+3}$$

+3

$$\frac{\sqrt{n}}{2n^2+3} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} \quad +2$$

so $\sum$ converges by comparison test with $\sum \frac{1}{n^{3/2}}$

+3

$\left(\sum \frac{1}{n^{3/2}}\right.$ converges since $3/2 > 1\left.\right)$ +2

(e)

$$\sum_{n=1}^{\infty}\left(\frac{2n+1}{n}\right)^{-n}$$

Root test:
+4

$$\sqrt[n]{\left(\frac{2n+1}{n}\right)^{-n}} = \left(\frac{2n+1}{n}\right)^{-1} \quad +2$$

$$= \frac{n}{2n+1} \xrightarrow[+1]{n\to\infty} \frac{1}{2}$$

$\frac{1}{2} < 1$ so $\sum$ converges +3

2. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$$

is conditionally convergent, absolutely convergent, or divergent.

Σ is not abs. convergent, since $\sum \frac{1}{\sqrt{n^2+1}}$ diverges +2 (~~[illegible]~~ L.C.T. with $\frac{1}{n}$) +3

(note comparison test doesn't work)

+2

Σ is convergent, by AST:

- $\frac{1}{\sqrt{n^2+1}} > 0$
- $\lim_{n \to \infty} \frac{1}{\sqrt{n^2+1}} = 0$ ✓ +1
- $\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{(n+1)^2+1}}$ ✓ +1

$\therefore \Sigma$ is conditionally convergent +1

3. Find the interval of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^{3/2}}$$

+2

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x-1)^{n+1}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(2x-1)^n}\right|$

$= \left|(2x-1)\left(\frac{n}{n+1}\right)^{3/2}\right|$

$= |2x-1|$ +2

$|2x-1| < 1$ if

$-1 < 2x-1 < 1$

$0 < 2x < 2$

$0 < x < 1$ +2

Test endpts: $x=0$: $\sum \frac{(-1)^n}{n^{3/2}}$ converges by AST
+1 +1

$x=1$: $\sum \frac{1}{n^{3/2}}$ converges since $3/2 > 1$
+1

~~4~~ (or: since $\sum$ converges abs. $(x=1)$ $\Rightarrow$ converges at $x=0$ also)

$\Rightarrow$ I.O.C. is $[0,1]$ +1

4. (a) Let $f(x) = \frac{1}{9+x^2}$. Represent f as a power series and find the interval of convergence for the power series.

(b) Represent f' as a power series. What is the radius of convergence of this series?

(c) Represent $g(x) = \frac{x}{9+(2x)^2}$ as a power series.

a) $$\frac{1}{9+x^2} = \frac{1}{9}\left(\frac{1}{1+\frac{x^2}{9}}\right) =$$

$$= \frac{1}{9}\left(\frac{1}{1-\left(-\frac{x^2}{9}\right)}\right) + 2$$

geo. series

$$= \frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n + 2$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{9}\left(\frac{x^{2n}}{9^n}\right)$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{9^{n+1}}$$

I.O.C.: Geo. series with ratio $\frac{x^2}{9}$, so converges only if $\left|\frac{x^2}{9}\right|<1$, i.e. if $|x|<3$. I.O.C. is $(-3,3)$ + 1

b) We can differentiate term-by-term:

$$f' = \left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{9^{n+1}}\right)' = \sum_{n=1}^{\infty}(-1)^n\frac{2nx^{2n-1}}{9^{n+1}}$$ + 2

R.O.C. = 3 (same as for f) + 1

c) $$\frac{x}{9+(2x)^2} = x\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{9^{n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{4^n x^{2n+1}}{9^{n+1}}$$ + 2

5. Find the Taylor series for $\cos(2x)$ centered at π.

$f(x) = \cos 2x$ $\qquad$ $f(\pi) = \cos 2\pi = 1$

$f'(x) = -2\sin 2x$ $\qquad$ $f'(\pi) = 0$

$f''(x) = -4\cos 2x$ $\qquad$ $f''(\pi) = -4$

$f'''(x) = 8\sin 2x$ $\qquad$ $f'''(\pi) = 0$

$f^{(4)}(x) = 16\cos 2x$ $\qquad$ $f^{(4)}(\pi) = 16$

$\dots$ $+2$ $\qquad$ $\dots$ $+2$

So Taylor series is

$$1 - \frac{4}{2!}(x-\pi)^2 + \frac{16}{4!}(x-\pi)^4 - \frac{2^6}{6!}(x-\pi)^6 + \dots \qquad +6$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!}(x-\pi)^{2n}$$