Test 1
July 14

Name:

1. Evaluate each of the following integrals. If any of them are divergent, show why.

   (a)

$$\int \csc^6 x \, dx$$

$1 + \cot^2 = \csc^2$

+3

$$= \int (1+\cot^2 x)^2 \csc^2 x \, dx$$

$u = \cot x$

$du = -\csc^2 x \, dx$

+3

$$= -\int (1+u^2)^2 \, du$$

+2

~~[illegible]~~

+2

$$= -\int 1 + 2u^2 + u^4 \, du$$

$$= -\left[u + \frac{2}{3}u^3 + \frac{u^5}{5} + C\right]$$

+1

$$= -\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C$$

(b)

$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx$$

$x = 3\sec\theta$ $\quad +2$

$dx = 3\sec\theta\tan\theta$

$$= \int \frac{\sqrt{(3\sec\theta)^2 - 9}}{(3\sec\theta)^3}\, 3\sec\theta\tan\theta\, d\theta \quad \leftarrow (\sec^2 - 1 = \tan^2)$$

$+1$

$$= \int \frac{3^2\tan^2\theta}{3^3\sec^2\theta}\, d\theta$$

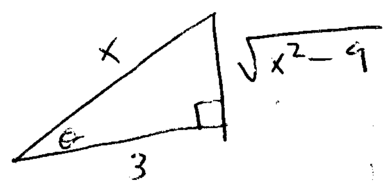
$$\left(\sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1}{3}\int \sin^2\theta\, d\theta \quad \leftarrow$$

$+2$

$$= \frac{1}{3}\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{6}\left(\theta - \frac{1}{2}\sin 2\theta + C\right) \quad +1$$




$\left(\sec\theta = \frac{x}{3}\right)$

$+2$

$$= \frac{1}{6}\theta - \frac{1}{12}\sin 2\theta + C = \frac{1}{6}\theta - \frac{1}{6}\sin\theta\cos\theta + C$$

$+1$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\left(\frac{\sqrt{x^2-9}}{x}\right)\left(\frac{3}{x}\right) + C$$

2

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C \quad +1$$

(c)

$$\int_0^3 \frac{dx}{x^2 - x - 2}$$

Improper! +3

$$\int_0^2 \frac{dx}{(x+1)(x-2)} + \int_2^3 \frac{dx}{(x+1)(x-2)}$$

Partial fractions

$$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

+3

$$1 = A(x-2) + B(x+1)$$

$x=2 \Rightarrow B = 1/3$

$x=-1 \Rightarrow A = -1/3$

$$\int_0^2 \frac{-1/3}{x+1} + \frac{1/3}{x-2}\,dx = \lim_{t\to 2^-} \int_0^t \frac{-1/3}{x+1} + \frac{1/3}{x-2}\,dx$$

+1

$$= \lim_{t\to 2^-} \left[\frac{1}{3}\left[-\ln|x+1| + \ln|x-2|\right]\right]_0^t$$

+1

3

$$= \lim_{t\to 2^-} \left[\frac{1}{3}\left(\ln|t-2| - \ln|t+1| - \ln 2 + \ln 1\right)\right]$$

$= -\infty$ (diverges) +2

(d)

$$\int_2^6 \frac{y}{\sqrt{y-2}}\, dy$$

+3

Improper (disc. @ 2)

$$= \lim_{t \to 2^+} \int_t^6 \frac{y}{\sqrt{y-2}}\, dy$$

+1

+2

$u = y-2$
$du = dy$

$$= \lim_{t \to 0^+} \int_t^4 \frac{u+2}{\sqrt{u}}\, du$$

(I think $u = \sqrt{y-2}$ works also)

$$= \lim_{t \to 0^+} \int_t^4 \sqrt{u} + \frac{2}{\sqrt{u}}\, du$$

$$= \lim_{t \to 0^+} \left[\frac{2}{3}u^{\frac{3}{2}} + 4u^{1/2} + C\right]_t^4$$ +2

$$= \lim_{t \to 0^+} \left[\frac{2}{3}4^{3/2} + 4\cdot\sqrt{4} - \frac{2}{3}t^{3/2} - 4\sqrt{t}\right]$$

+2

$$= \frac{16}{3} + 8 = \frac{40}{3}$$

2. Find the arc length of the curve $y = \ln(\sec x)$, where $0 \le x \le \pi/4$

$y = \ln(\sec x)$

$\frac{dy}{dx} = \frac{1}{\sec x} \sec x \tan x \, dx = \tan x \, dx$ +2

Arc length $= \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx$ +4

$= \int_0^{\frac{\pi}{4}} \sec x$ +1

$= \left[ \ln|\sec x + \tan x| \right]_0^{\pi/4}$ +2

$= \ln(\sqrt{2}+1) - \ln 1$

$= \ln(\sqrt{2}+1)$ +1

3. Find the area of the surface obtained by rotating the curve $y = 1 - x^2, 0 \le x \le 1$ about the $x$-axis.

$\frac{dy}{dx} = -2x$ +2

$SA = 2\pi \int_0^1 (1-x^2)\sqrt{1+4x^2}\,dx$    $u = 2x$, $du = 2dx$    + 7

$= \frac{2\pi}{2} \int \left(1 - \frac{u^2}{4}\right)\sqrt{1+u^2}\,du$    +1

$= \pi \int \left(1 - \frac{\tan^2\theta}{4}\right) \sec^3\theta\,d\theta$    $u = \tan\theta$, $du = \sec^2\theta\,d\theta$

$= \pi \int \sec^3\theta - \frac{\sin^2\theta}{4\cos^5\theta}\,d\theta$

4. For which values of $p$ does the following integral converge? Evaluate the integral for those values.

$$\int_1^\infty \frac{\ln x}{x^p}\,dx$$

$$\int_1^\infty \frac{\ln x}{x^p} = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^p}$$

$u = \ln x$ $\quad v = \frac{x^{-p+1}}{-p+1}$

$du = \frac{1}{x}dx$ $\quad dv = x^{-p}$

$$= \lim_{t\to\infty} \left[\frac{\ln x\; x^{-p+1}}{-p+1}\right]_1^t - \int_1^t \frac{x^{-p}}{p+1}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{1-p}\left(\frac{\ln t}{t^{p-1}}\right) - 0\right] - \left[\frac{1}{(1-p)^2}x^{-p+1}\right]_1^t$$

$+5$

$p > 1$ $$= \lim_{t\to\infty} \left[\frac{1}{1-p}\left(\frac{1}{(p-1)t^{p-1}}\right) - \frac{1}{(1-p)^2}t^{1-p} + 1\right]$$

$$= \frac{1}{(1-p)^2}$$

$+2$

$p < 1$ $\quad = \infty$

$+2$

$p = 1$ $$\int_1^\infty \ln x = \lim_{t\to\infty} \int_1^t \ln x$$

$u = \ln x$ $\quad v = x$

$du = \frac{1}{x}dx$ $\quad dv = dx$

$$= \lim_{t\to\infty} \left[x \ln x\Big|_1^t - \int_1^t dx\right]$$

$+1$

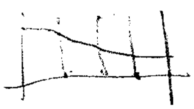
$$= \lim_{t\to\infty} t\ln t - 0 - t + 1$$

$$= \lim_{t\to\infty} t(\ln t - 1) + 1 = \infty$$

5. (a) Write the approximations $L_4$, $M_4$, and $S_4$ for

$$\int_0^1 e^{-x^2/2}\,dx$$

Do not evaluate.



(b) Is $L_4$ an over-estimate or under-estimate for the integral? Explain.

(c) How large must $n$ be to guarantee $M_n$ is within $10^{-5}$ of the true value?

a) $L_4 = \sum_{i=0}^{3} \frac{1}{4}\left(e^{-\left(\frac{i}{4}\right)^2/2}\right) = \frac{1}{4}\left[e^{-0/2} + e^{-1/4/2} + e^{-1/2/2} + e^{-3/4/2}\right]$ + 2

$M_4 = \sum_{i=0}^{3} \frac{1}{4}\left(e^{-\left(\frac{1}{8}+\frac{i}{4}\right)^2/2}\right)$ +2

$S_4 = \frac{1}{12}\left[e^{-0/2} + 4e^{-(1/4)^2/2} + 2e^{-(1/2)^2/2} + 4e^{-(3/4)^2/2} + e^{-1^2/2}\right]$ +2

b) $L_4$ over-estimate since $e^{-x^2/2}$ decreasing on $[0,1]$ +2

c) $f = e^{-x^2/2}$

$f' = -xe^{-x^2/2}$

$f'' = -e^{-x^2/2} + (-x)(-x)e^{-x^2/2}$

$= (x^2-1)e^{-x^2/2}$

$|f''| \leq 1$

$|E_M| \leq \frac{1}{24n^2} \leq 10^{-5}$

$\Rightarrow n^2 \geq \frac{10^5}{24}$ +2

$n \geq \sqrt{\frac{10^5}{24}}$