Test 0 Solutions

June 30

Name:

1. Evaluate the following integral:

$$\int_1^{e^3} \frac{(\ln x)^2}{x} dx =$$

If we let $u = \ln x$, then $du = \frac{1}{x} dx$, and

$$\begin{aligned}
\int_1^{e^3} \frac{(\ln x)^2}{x} dx &= \int_0^3 u^2 du \\
&= \left[\frac{u^3}{3}\right]_0^3 \\
&= 9 - 0 = 9
\end{aligned}$$

2. Find an antiderivative:

$$\int \sin(\pi t)\cos(\pi t)dt =$$

If we let $u = \sin(\pi t)$, then $du = \pi\cos(\pi t)dt$, and

$$\begin{aligned}\int \sin(\pi t)\cos(\pi t)dt &= \int \frac{u}{\pi}du \\ &= \frac{u^2}{2\pi} + C \\ &= \frac{\sin^2(\pi t)}{2\pi} + C\end{aligned}$$

3. Put the following quantities in order, from least to greatest. Briefly explain your choices.

$$A = \int_0^{\pi/2} \cos(t)dt$$
$$B = \int_0^{\pi} \cos(t)dt$$
$$C = \int_0^{3} \cos(t)dt$$
$$D = \int_0^{6} \cos(t)dt$$
$$E = \int_0^{3\pi/2} \cos(t)dt$$

E, D, B, C, A

4. Let

$$K = \int_{-9}^{4} 3x^2 dx$$

(a) Give an approximation for K with 39 rectangles and left endpoints.

(b) Is your approximation an underestimate or an overestimate? Explain why.

(c) Express K as a limit of sums.

(a)

$$\sum_{i=0}^{38} \frac{1}{3} \cdot 3(-9 + \frac{1}{3}i)^2$$

(b) The approximation is an overestimate. We are overestimating from -9 to 0, and underestimating from 0 to 4, but overall we are overestimating.

(c)

$$K = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{13}{n} \cdot 3(-9 + \frac{13}{n}i)^2$$

5. Do two of the following four problems:

(a) Find all points on the graph of the function

$$f(x) = 2\sin(x) + \sin^2(x)$$

at which the tangent line is horizontal.

(b) Find the derivative of the function $f(x) = x^3$ at $x = 1$ directly from the definition.

(c) Evaluate

$$\lim_{x\to\pi} \frac{e^{\sin x} - 1}{x - \pi}$$

(d) Find an equation of the tangent line to the curve $y = e^x \cos x$ at the point $(0, 1)$.

(a) The points where the tangent line is horizontal are the points where $f'(x) = 0$, i.e. where

$$0 = 2\cos(x) + 2\sin(x)\cos(x) = 2\cos(x)[1 + \sin(x)]$$

Thus the tangent line is horizontal when $2\cos(x) = 0$ (at $x = \frac{\pi}{2} + \pi n$) or when $\sin(x) = -1$ (at $x = \frac{3\pi}{2} + 2\pi n$).

(b)

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{(1+h)^3 - 1^3}{h} \\
&= \lim_{h\to 0} \frac{1^3 + 3\cdot 1^2 \cdot h + 3 \cdot 1 \cdot h^2 + h^3 - 1}{h} \\
&= \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{h} \\
&= \lim_{h\to 0} 3 + 3h + h^2 \\
&= 3
\end{aligned}$$

(c) Use L'Hopital's Rule:

$$\begin{aligned}
\lim_{x\to\pi} \frac{e^{\sin x} - 1}{x - \pi} &= \lim_{x\to\pi} \frac{\cos(x) e^{\sin x}}{1} \\
&= \cos \pi e^{\sin \pi} \\
&= -1 \cdot 1 = -1
\end{aligned}$$

(d) By the product rule,

$$y'(0) = -e^0 \sin(0) + \cos(0) e^0 = 0 + 1 = 1$$

Thus the tangent line passes through $(0, 1)$ and has slope 1. An equation for this line is $y = x + 1$.

6. Let f be a differentiable, non-constant, function with the property that

$$\int_0^x f(t)dt = [f(x)]^2$$

What function is f?

If we differentiate the integral, by the Fundamental Theorem, we get f(x). Thus differentiating both sides with respect to x, we get

$$f(x) = 2f(x)f'(x)$$

Thus $0 = f(x)[2f'(x) - 1]$, which implies $f(x) = 0$ or $f'(x) = 1/2$ for all x. Since f is non-constant, it can't be identically 0, and thus $f(x) = x/2 + C$. Plugging back into the original equation, we find $C = 0$, and so conclude $f(x) = x/2$.