Quiz 7 Solutions

August 4

Name:

1. Solve the differential equation:

$$y' = xe^{-\sin x} - y\cos x$$

Writing this as

$$y' + y\cos x = xe^{-\sin x}$$

we see that the equation is a first order linear diff. eq. with $P(x) = \cos x$. Thus

$$I(x) = e^{\int \cos x \ dx} = e^{\sin x}$$

Multiplyting through by $I(x)$, we get

$$(ye^{\sin x})' = x$$

Then we antidifferentiate to get

$$ye^{\sin x} = x^2/2 + C$$

or

$$y = (x^2/2 + C)e^{-\sin x}$$

2. Solve the differential equation:

$$xyy' = \ln x \quad , \quad y(1) = 2$$

This is a separable diff. eq. (with no equilibrium solutions), so we can write

$$\int y \; dy = \int \frac{\ln x}{x} \; dx$$

and antidifferentiate to get

$$y^2/2 = (\ln x)^2/2 + C$$

Plugging in the initial condition $y(1) = 2$ we get $2 = C$. Thus

$$y = \sqrt{(\ln x)^2 + 4}$$

3. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed, and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

Let $y(t)$ = amount of salt (in kg) in the tank after t minutes.

Then

$$\begin{aligned}\frac{dy}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= (0.1)(10) - \left(\frac{y(t)}{100}\right)(10) \\ &= 1 - \frac{y}{10} \\ &= \frac{10-y}{10}\end{aligned}$$

This is a separable diff. eq. (with equilibrium solution $y = 10$), so we write

$$\int \frac{dy}{10-y} = \int \frac{dt}{10}$$

Antidifferentiating, we get

$$-\ln|10-y| = t/10 + C$$

Plug in the initial condition $y(0) = 0$ to get $-\ln 10 = C$, and do some algebra to get the explicit solution

$$\ln|10-y| = \ln 10 - t/10$$

$$|10-y| = e^{\ln 10}e^{-t/10} = 10e^{-t/10}$$

Since $y < 10$, we may drop the absolute value symbols:

$$y = 10 - 10e^{-t/10} = 10(1-e^{-t/10})$$

After six minutes,the quantity of salt in the tank will be

$$y(6) = 10(1-e^{-6/10}) = 10(1-e^{-3/5}) \text{ kg.}$$

4. Answer question 3 if the tank drains at the rate of 5 L/min.

   Again, let $y(t)$ = amount of salt (in kg) in the tank after t minutes. Then

$$\begin{aligned}\frac{dy}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= (0.1)(10) - \left(\frac{y(t)}{100+5t}\right)(5) \\ &= 1 - \frac{5y}{100+5t}\end{aligned}$$

   Writing this as

$$y' + \frac{5y}{100+5t} = 1$$

   we see that this is a linear diff. eq. with $P(t) = \frac{5}{100+5t}$. Thus

$$I(t) = e^{\int \frac{5}{100+5t}\ dx} = e^{\ln|100+5t|} = (100+5t)$$

   Multiplying through by $I(t)$, we get

$$((100+5t)y)' = (100+5t)$$

   Antidifferentiating, we find

$$(100+5t)y = \frac{1}{10}(100+5t)^2 + C$$

   Plugging in the initial condition $y(0) = 0$, we find

$$C = -\frac{100^2}{10} = -1000$$

   and thus

$$y = \frac{100+5t}{10} - \frac{1000}{(100+5t)}$$

   After six minutes, the quantity of salt in the tank will be

$$y(6) = \frac{130}{10} - \frac{1000}{(130)} = 13 - \frac{100}{13}\ \text{kg.}$$