4. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. $\int_1^\infty x^{-1/4}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/4}\,dx = \lim_{t\to\infty} \left[\frac{4}{3}x^{3/4}\right]_1^t = \lim_{t\to\infty}\left(\frac{4}{3}t^{3/4} - \frac{4}{3}\right) = \infty$, so $\sum_{n=1}^\infty 1/\sqrt[4]{n}$ diverges.

8. The function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \left(1 + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} [x + \ln(x+1)]_1^t = \lim_{t\to\infty} (t + \ln(t+1) - 1 - \ln 2) = \infty,$$ so $\displaystyle\int_1^\infty \frac{x+2}{x+1}\,dx$ is divergent and the series $\displaystyle\sum_{n=1}^\infty \frac{n+2}{n+1}$ is divergent. NOTE: $\displaystyle\lim_{n\to\infty} \frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

18. The function $f(x) = \dfrac{1}{x^2 - 4x + 5} = \dfrac{1}{(x-2)^2 + 1}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies. $\displaystyle\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x-2)^2+1}\,dx = \lim_{t\to\infty}[\tan^{-1}(x-2)]_2^t =$ $\displaystyle\lim_{t\to\infty}[\tan^{-1}(t-2) - \tan^{-1} 0] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$, so the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n^2 - 4n + 5}$ converges. Of course this means that $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2 - 4n + 5}$ converges too.

20. $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive for $x \geq 2$, and $f'(x) = \dfrac{1 - 2\ln x}{x^3} < 0$ for $x \geq 2$, so f is decreasing. $\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^t$ [by parts] $\overset{H}{=} 1$. Thus, $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2} = \sum_{n=2}^\infty \frac{\ln n}{n^2}$ converges by the Integral Test.

27. Clearly the series cannot converge if $p \geq -\frac{1}{2}$, because then $\lim_{n\to\infty} n(1+n^2)^p \neq 0$. Also, if $p = -1$ the series diverges (see Exercise 17). So assume $p < -\frac{1}{2}$, $p \neq -1$. Then $f(x) = x(1+x^2)^p$ is continuous, positive, and eventually decreasing on $[1, \infty)$, and we can use the Integral Test.
$\displaystyle\int_1^\infty x(1+x^2)^p dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+x^2)^{p+1}}{p+1}\right]_1^t = \lim_{t\to\infty}\frac{1}{2}\cdot\frac{(1+t^2)^{p+1}}{p+1} - \frac{2^p}{p+1}$. This limit exists and is finite $\Leftrightarrow\ p + 1 < 0\ \Leftrightarrow\ p < -1$, so the series converges whenever $p < -1$.

4. $\dfrac{2}{n^3+4} < \dfrac{2}{n^3}$ for all $n \geq 1$, so $\displaystyle\sum_{n=1}^\infty \frac{2}{n^3+4}$ converges by comparison with $\displaystyle\sum_{n=1}^\infty \frac{2}{n^3} = 2\sum_{n=1}^\infty \frac{1}{n^3}$, which converges because it is a constant multiple of a convergent p-series ($p = 3 > 1$).

10. $\dfrac{n^2-1}{3n^4+1} < \dfrac{n^2}{3n^4+1} < \dfrac{n^2}{3n^4} = \dfrac{1}{3}\dfrac{1}{n^2}$. $\displaystyle\sum_{n=1}^\infty \frac{n^2-1}{3n^4+1}$ converges by comparison with $\displaystyle\sum_{n=1}^\infty \frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series ($p = 2 > 1$). The terms of the given series are positive for $n > 1$, which is good enough.

16. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ $\left(p = \frac{3}{2} > 1\right)$.

[illegible] $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdots\cdot 1$ for $n \geq 2$, so since $\displaystyle\sum_{n=1}^\infty \frac{2}{n^2}$ converges ($p = 2 > 1$), $\displaystyle\sum_{n=1}^\infty \frac{n!}{n^n}$ converges also by the Comparison Test.

14. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\left(\dfrac{\ln n}{n}\right) = 0 + \sum\limits_{n=2}^{\infty}(-1)^{n-1}\left(\dfrac{\ln n}{n}\right)$. $b_n = \dfrac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \dfrac{\ln x}{x}$, then $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

18. $\sum\limits_{n=1}^{\infty}(-1)^n \cos\left(\dfrac{\pi}{n}\right)$. $\lim\limits_{n\to\infty} \cos\left(\dfrac{\pi}{n}\right) = \cos(0) = 1$, so $\lim\limits_{n\to\infty} (-1)^n \cos\left(\dfrac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

24. The series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)^4} < \dfrac{1}{n^4}$ and

(ii) $\lim\limits_{n\to\infty} \dfrac{1}{n^4} = 0$, so the series is convergent. Now $b_5 = 1/5^4 = 0.0016 > 0.001$ and

$b_6 = 1/6^4 \approx 0.00077 < 0.001$, so by the Alternating Series Estimation Theorem, $n = 5$.

28. $b_6 = \dfrac{6}{8^6} = \dfrac{6}{262{,}144} \approx 0.000\,023$, so

$\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n n}{8^n} \approx s_5 = \sum\limits_{n=1}^{5} \dfrac{(-1)^n n}{8^n} = -\frac{1}{8} + \frac{2}{64} - \frac{3}{512} + \frac{4}{4096} - \frac{5}{32{,}768} \approx -0.098\,785$. Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.0988.

32. If $p > 0$, $\dfrac{1}{(n+1)^p} \leq \dfrac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \leq 0$, $\lim\limits_{n\to\infty} \dfrac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

34. Let $f(x) = \dfrac{(\ln x)^p}{x}$. Then $f'(x) = \dfrac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p. Clearly $\lim\limits_{n\to\infty} \dfrac{(\ln n)^p}{n} = 0$ if $p \leq 0$, and if $p > 0$ we can apply l'Hospital's Rule $[\![p+1]\!]$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).