10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim\limits_{x\to-\infty} \int_x^{-1} e^{-2t}\,dt = \lim\limits_{x\to-\infty} \left[-\frac{1}{2}e^{-2t}\right]_x^{-1} = \lim\limits_{x\to-\infty} \left[-\frac{1}{2}e^{2} + \frac{1}{2}e^{-2x}\right] = \infty.$ Divergent

14. $\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx = \int_{-\infty}^{0} x^2 e^{-x^3}\,dx + \int_0^{\infty} x^2 e^{-x^3}\,dx$, and

$\int_{-\infty}^{0} x^2 e^{-x^3}\,dx = \lim\limits_{t\to-\infty} \left[-\frac{1}{3}e^{-x^3}\right]_t^0 = -\frac{1}{3} + \frac{1}{3}\left(\lim\limits_{t\to-\infty} e^{-t^3}\right) = \infty.$ Divergent

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to1^-} \left[\sin^{-1} x\right]_0^t = \lim_{t\to1^-} \sin^{-1} t = \frac{\pi}{2}.$ Convergent

36. $\displaystyle\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \frac{dx}{(x+3)(x-2)} = \int_0^2 \frac{dx}{(x-2)(x+3)} + \int_2^4 \frac{dx}{(x-2)(x+3)}$, and

$$\int_0^2 \frac{dx}{(x-2)(x+3)} = \lim_{t\to2^-} \int_0^t \left[\frac{1/5}{x-2} - \frac{1/5}{x+3}\right] dx \quad \text{[partial fractions]} \quad = \lim_{t\to2^-} \left[\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|\right]_0^t$$

$$= \lim_{t\to2^-} \frac{1}{5}\left[\ln\left|\frac{t-2}{t+3}\right| - \ln\frac{2}{3}\right] = -\infty. \text{ Divergent}$$

58. Let $u = \ln x$. Then $du = dx/x \;\Rightarrow\; \displaystyle\int_e^{\infty} \frac{dx}{x(\ln x)^p} = \int_1^{\infty} \frac{du}{u^p}$. By Example 4, this converges to $\dfrac{1}{p-1}$ if $p > 1$ and diverges otherwise.

6. $y^2 = 4(x+4)^3,\, y > 0 \;\Rightarrow\; y = 2(x+4)^{3/2} \;\Rightarrow\; dy/dx = 3(x+4)^{1/2} \;\Rightarrow$
$1 + (dy/dx)^2 = 1 + 9(x+4) = 9x + 37$. So

$$L = \int_0^2 \sqrt{9x+37}\,dx \quad \begin{bmatrix} u = 9x+37, \\ du = 9\,dx \end{bmatrix} \quad = \int_{37}^{55} u^{1/2}\left(\frac{1}{9}\,du\right)$$

$$= \frac{1}{9}\cdot\frac{2}{3}\left[u^{3/2}\right]_{37}^{55} = \frac{2}{27}\left(55\sqrt{55} - 37\sqrt{37}\right)$$

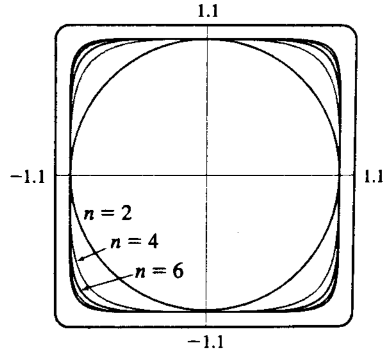
10. $y = \ln(\cos x) \;\Rightarrow\; dy/dx = -\tan x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \tan^2 x = \sec^2 x$. So
$L = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/3} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/3} = \ln(2+\sqrt{3}) - \ln(1+0) = \ln(2+\sqrt{3})$.

14. $y^2 = 4x,\, x = \frac{1}{4}y^2 \;\Rightarrow\; \frac{dx}{dy} = \frac{1}{2}y \;\Rightarrow\; 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y^2$. So

$$L = \int_0^2 \sqrt{1 + \tfrac{1}{4}y^2}\,dy = \int_0^1 \sqrt{1+u^2}\cdot 2\,du \qquad [u = \tfrac{1}{2}y,\, dy = 2\,du]$$

$$\overset{21}{=} \left[u\sqrt{1+u^2} + \ln\left|u + \sqrt{1+u^2}\right|\right]_0^1 = \sqrt{2} + \ln(1+\sqrt{2})$$

38. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4.



$x^{2k} + y^{2k} = 1 \;\Rightarrow\; y^{2k} = 1 - x^{2k} \;\Rightarrow\; y = \left(1 - x^{2k}\right)^{1/(2k)}$

(in the first quadrant), so we use the arc length formula with

$$\frac{dy}{dx} = \frac{1}{2k}\left(1 - x^{2k}\right)^{1/(2k)-1}\left(-2kx^{2k-1}\right)$$
$$= -x^{2k-1}\left(1 - x^{2k}\right)^{1/(2k)-1}$$

The total length is therefore

$$L_{2k} = 4\int_0^1 \sqrt{1 + \left[-x^{2k-1}\left(1 - x^{2k}\right)^{1/(2k)-1}\right]^2}\,dx = 4\int_0^1 \sqrt{1 + x^{2(2k-1)}\left(1 - x^{2k}\right)^{1/k-2}}\,dx$$

Now from the graph, we see that as k increases, the "corners" of these fat circles get closer to the points $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$, and the "edges" of the fat circles approach the lines joining these four points. It seems plausible that as $k \to \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \to \infty$ of the equation of the fat circle in the first quadrant: $\lim\limits_{k\to\infty}\left(1 - x^{2k}\right)^{1/(2k)} = 1$ for $0 \le x < 1$. So we guess that $\lim\limits_{k\to\infty} L_{2k} = 4 \cdot 2 = 8$.

8. $y = \cos 2x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (-2\sin 2x)^2}\,dx \;\Rightarrow$

$S = \int_0^{\pi/6} 2\pi \cos 2x \sqrt{1 + 4\sin^2 2x}\,dx = 2\pi \int_0^{\sqrt{3}} \sqrt{1 + u^2}\left(\frac{1}{4}\,du\right) \qquad [u = 2\sin 2x,\ du = 4\cos 2x\,dx]$

$\overset{21}{=} \frac{\pi}{2}\left[\frac{1}{2}u\sqrt{1 + u^2} + \frac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^{\sqrt{3}} = \frac{\pi}{2}\left[\frac{\sqrt{3}}{2} \cdot 2 + \frac{1}{2}\ln\left(\sqrt{3} + 2\right)\right] = \frac{\pi\sqrt{3}}{2} + \frac{\pi}{4}\ln\left(2 + \sqrt{3}\right)$

25. $S = 2\pi \int_1^\infty y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx = 2\pi \int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3}\,dx$. Rather than trying to evaluate this integral, note that $\sqrt{x^4 + 1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$$S = 2\pi \int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3}\,dx > 2\pi \int_1^\infty \frac{x^2}{x^3}\,dx = 2\pi \int_1^\infty \frac{1}{x}\,dx$$

But we know that this integral diverges, so the area S is infinite.

29. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\Rightarrow\quad \frac{y\,(dy/dx)}{b^2}=-\frac{x}{a^2} \quad\Rightarrow\quad \frac{dy}{dx}=-\frac{b^2x}{a^2y} \quad\Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{b^4x^2}{a^4y^2}=\frac{b^4x^2+a^4y^2}{a^4y^2}=\frac{b^4x^2+a^4b^2\,(1-x^2/a^2)}{a^4b^2\,(1-x^2/a^2)}=\frac{a^4b^2+b^4x^2-a^2b^2x^2}{a^4b^2-a^2b^2x^2}$$

$$=\frac{a^4+b^2x^2-a^2x^2}{a^4-a^2x^2}=\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}$$

The ellipsoid's surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x-axis. Thus,

$$S=2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx$$

$$=\frac{4\pi b}{a^2}\int_0^a \sqrt{a^4-(a^2-b^2)x^2}\,dx=\frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}} \sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}} \qquad [u=\sqrt{a^2-b^2}\,x]$$

$$\overset{30}{=}\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2}+\frac{a^4}{2}\sin^{-1}\frac{u}{a^2}\right]_0^{a\sqrt{a^2-b^2}}$$

$$=\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4-a^2(a^2-b^2)}+\frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]=2\pi\left[b^2+\frac{a^2b\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$