10. $\int_0^\pi \cos^6\theta\,d\theta = \int_0^\pi (\cos^2\theta)^3\,d\theta = \int_0^\pi \left[\frac{1}{2}(1+\cos 2\theta)\right]^3 d\theta = \frac{1}{8}\int_0^\pi (1+3\cos 2\theta + 3\cos^2 2\theta + \cos^3 2\theta)\,d\theta$

$= \frac{1}{8}\left[\theta + \frac{3}{2}\sin 2\theta\right]_0^\pi + \frac{1}{8}\int_0^\pi \left[\frac{3}{2}(1+\cos 4\theta)\right]d\theta + \frac{1}{8}\int_0^\pi \left[(1-\sin^2 2\theta)\cos 2\theta\right]d\theta$

$= \frac{1}{8}\pi + \frac{3}{16}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^\pi + \frac{1}{8}\int_0^0 (1-u^2)\left(\frac{1}{2}du\right) \quad [u = \sin 2\theta,\ du = 2\cos 2\theta\,d\theta]$

$= \frac{\pi}{8} + \frac{3\pi}{16} + 0 = \frac{5\pi}{16}$

24. $\int \tan^4 x\,dx = \int \tan^2 x\,(\sec^2 x - 1)\,dx = \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C$
(Set $u = \tan x$ in the first integral and use Exercise 23 for the second.)

36. $\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \int_{\pi/4}^{\pi/2} \cot x\,(\csc^2 x - 1)\,dx = \int_{\pi/4}^{\pi/2} \cot x \csc^2 x\,dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

$= \left[-\frac{1}{2}\cot^2 x - \ln|\sin x|\right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\frac{1}{2} - \ln\frac{1}{\sqrt{2}}\right] = \frac{1}{2} + \ln\frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \ln 2)$

46. $\displaystyle\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1}\cdot\frac{\cos x + 1}{\cos x + 1}\,dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx = \int \frac{\cos x + 1}{-\sin^2 x}\,dx$

$= \int \left(-\cot x \csc x - \csc^2 x\right)dx = \csc x + \cot x + C$

53. $f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x \cos^3 x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\,(1 - \sin^2 x)\cos x\,dx$

$= \frac{1}{2\pi}\int_0^0 u^2(1-u^2)\,du \qquad [\text{where } u = \sin x]$

$= 0$

4. Let $x = 4\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 4\cos\theta\,d\theta$ and $\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4|\cos\theta| = 4\cos\theta$. When $x = 0$, $4\sin\theta = 0 \Rightarrow \theta = 0$, and when $x = 2\sqrt{3}$, $4\sin\theta = 2\sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$. Thus, substitution gives

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\cos\theta}4\cos\theta\,d\theta = 4^3\int_0^{\pi/3}\sin^3\theta\,d\theta$$

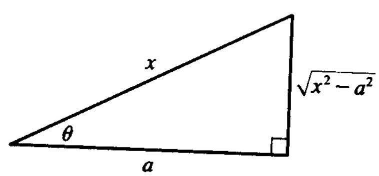
$$= 4^3\int_0^{\pi/3}(1-\cos^2\theta)\sin\theta\,d\theta$$

$$\overset{c}{=} -4^3\int_1^{1/2}(1-u^2)\,du = -64\left[u - \tfrac{1}{3}u^3\right]_1^{1/2}$$

$$= -64\left[\left(\tfrac{1}{2} - \tfrac{1}{24}\right) - \left(1 - \tfrac{1}{3}\right)\right] = -64\left(-\tfrac{5}{24}\right) = \tfrac{40}{3}$$

Or: Let $u = 16 - x^2$, $x^2 = 16 - u$, $du = -2x\,dx$.

8. Let $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$, so

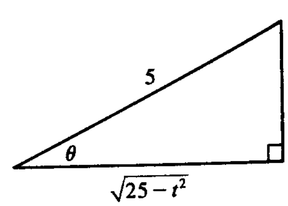


$$\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \int \frac{a\tan\theta}{a^4\sec^4\theta}a\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{a^2}\int \sin^2\theta\cos\theta\,d\theta$$

$$= \frac{1}{3a^2}\sin^3\theta + C = \frac{(x^2-a^2)^{3/2}}{3a^2x^3} + C$$

20. Let $t = 5\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dt = 5\cos\theta\,d\theta$ and $\sqrt{25 - t^2} = 5\cos\theta$, so

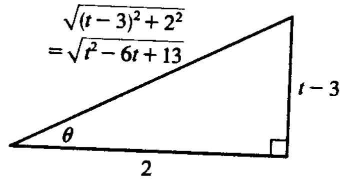


$$\int \frac{t}{\sqrt{25-t^2}}\,dt = \int \frac{5\sin\theta}{5\cos\theta}5\cos\theta\,d\theta = 5\int \sin\theta\,d\theta$$

$$= -5\cos\theta + C = -5\cdot\frac{\sqrt{25-t^2}}{5} + C = -\sqrt{25-t^2} + C$$

Or: Let $u = 25 - t^2$, so $du = -2t\,dt$.

24. $t^2 - 6t + 13 = (t^2 - 6t + 9) + 4 = (t-3)^2 + 2^2$. Let $t - 3 = 2\tan\theta$, so $dt = 2\sec^2\theta\, d\theta$. Then



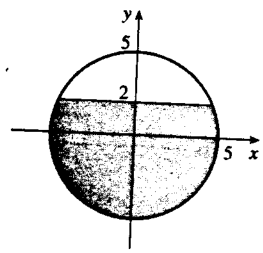
$$\int \frac{dt}{\sqrt{t^2 - 6t + 13}} = \int \frac{1}{\sqrt{(2\tan\theta)^2 + 2^2}}\, 2\sec^2\theta\, d\theta = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C_1 \quad \text{[by Formula 7.2.1]}$$

$$= \ln\left|\frac{\sqrt{t^2 - 6t + 13}}{2} + \frac{t-3}{2}\right| + C_1$$

$$= \ln\left|\sqrt{t^2 - 6t + 13} + t - 3\right| + C \quad \text{where } C = C_1 - \ln 2$$

40. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is



$$A = 2\int_{-5}^{2} \sqrt{25 - y^2}\, dy$$

$$= \left[25\arcsin(y/5) + y\sqrt{25 - y^2}\right]_{-5}^{2} \quad \text{[substitute } y = 5\sin\theta\text{]}$$

$$= 25\arcsin\tfrac{2}{5} + 2\sqrt{21} + \tfrac{25}{2}\pi \approx 58.72 \text{ ft}^2$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2} \approx \dfrac{58.72}{25\pi} \approx 0.748$ or 74.8%.