# RANDOMIZED GREEDY MATCHING II

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**Abstract**

We consider the following randomized algorithm for finding a matching $M$ in an *arbitrary* graph $G = (V, E)$. Repeatedly, choose a random vertex $u$, then a random neighbour $v$ of $u$. Add edge $\{u, v\}$ to $M$ and delete vertices $u, v$ from $G$ along with any vertices that become isolated. Our main result is that there exists a positive constant $\epsilon$ such that the expected ratio of the size of the matching produced to the size of largest matching in $G$ is at least $.5 + \epsilon$. We obtain stronger results for sparse graphs and trees and consider extensions to hypergraphs.

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# 1 Introduction

Heuristic algorithms in Combinatorial Optimization often allow opportunities for randomization. An important question therefore is whether this can give measurable improvement. In this paper we continue a discussion of Greedy Matching algorithms initiated by Dyer and Frieze [2]. Consider the following heuristic for finding a large matching in a graph $G$:

**GREEDY MATCHING**

**begin**

$M \leftarrow \emptyset$; $\Gamma \leftarrow G$;

**while** $E(\Gamma) \neq \emptyset$ **do**

**begin**

$\boldsymbol{A}$: Choose $e = \{u, v\} \in E(\Gamma)$

$\Gamma \leftarrow \Gamma \setminus \{u, v\}$;

$M \leftarrow M \cup \{e\}$

**end**;

Output $M$

**end**

The choice of $e$ in statement $\boldsymbol{A}$ is unspecified. It is known [4] that, if the worst possible choices are made in $\boldsymbol{A}$, the size of the matching $M$ produced is at least one half of the size of the largest matching, and one half is attainable. (Consider choosing the middle edge of a path of length three.)

The choice rule in [2] was that $e$ was to be chosen randomly from $E(\Gamma)$. Let us denote this algorithm by Randomized Greedy (RG). The main results of that paper can be summarised as follows:

**NOTATION**

Let $G = (V, E)$ be a (simple) graph with $|V| = n$. For any $v \in V$, $N_G(v)$ denotes its neighbours in $G$. For any $S \subseteq V$, $G \setminus S$ denotes the subgraph induced by the vertex set $V \setminus S$. Let $m(G)$ be the maximum size of a matching in $G$ and let $\mu_0(G)$ be the expected size of the matching produced by RG.

Let

$$r_0(G) = \begin{cases} \mu_0(G)/m(G) & \text{if } m(G) > 0 \\ 1 & \text{if } m(G) = 0. \end{cases}$$

If $\mathcal{K}$ is any class of graphs $\rho_0(\mathcal{K}) = \inf_{G \in \mathcal{K}} r_0(G)$. Unless otherwise stated, $\mathcal{G}$ will denote any class of graphs closed under vertex deletions and (to avoid trivialities) we suppose $|E| > 0$ for some $G \in \mathcal{G}$.

$$\kappa_0(\mathcal{G}) = \inf_{G \in \mathcal{G}} \left\{ \frac{|V|}{2|E|} : G = (V, E),\ |E| > 0 \right\}$$

Note that since some $G \in \mathcal{G}$ has an edge, and $\mathcal{G}$ is closed under deletions, the graph containing a single edge lies in $\mathcal{G}$. Thus $0 \leq \kappa_0(\mathcal{G}) \leq 1$ for any $\mathcal{G}$. In particular $\kappa_0(\text{GRAPHS}) = 0$, $\kappa_0(\text{PLANAR GRAPHS}) = \frac{1}{6}$, $\kappa_0(\text{FORESTS}) = \frac{1}{2}$.

**Theorem 1** *(a)* $\rho_0(\mathcal{G}) \geq (2 - \kappa_0(\mathcal{G}))^{-1}$.

*(b)* $\rho_0(\text{GRAPHS}) = \frac{1}{2}$.

*(c)* $\rho_0(\text{FORESTS}) = \frac{2}{3} + 2 \sum_{k=0}^{\infty} \frac{(-2)^k}{(2k+5)!!} = 0.7690397\cdots$

*(where* $n!! = n(n-2)(n-4)\cdots 3 \cdot 1$ *for* $n$ *odd).*

There are obvious ways to improve RG without significantly increasing its complexity. We consider one such way here. We will change the choice rule to the following:

> Choose $u$ randomly from $V(\Gamma)$. If $N_\Gamma(u) \neq \emptyset$ then pick $v$ randomly from $N_\Gamma(u)$ and remove $u, v$ from $\Gamma$; otherwise, simply remove $u$ from $\Gamma$. (1)

We will call this algorithm Modified Randomized Greedy (MRG). (Its performance on random graphs and trees was determined in Dyer, Frieze and Pittel [3].)

MRG generally seems to have a better worst-case performance than RG. We have several results that support this statement. Examination of bad examples for RG gives some idea why. Let $G_m$ be the graph obtained by adding a new vertex and edge adjacent to each vertex of the complete graph $K_m$. It was shown in [2] that $r_0(G_m) = \frac{1}{2} + o(1)$. This is because **whp** i.e. with probability 1-o(1), most edge choices are from the $K_m$ instead of from the pendant edges. MRG will do better because it has a 50-50 chance of choosing a vertex of degree one.

Now let $\mu_1(G)$ be the expected size of the matching produced by MRG when run on a graph $G$. Let $r_1, \rho_1$ be defined for MRG analogously to the definition of $r_0, \rho_0$ for RG. Let vertex $v$ have degree $d_v$ and assume that $d_v > 0$ for all $v \in V$. Let

$$\kappa_1(G) = \frac{1}{n} \sum_{v \in V} \frac{1}{d_v}.$$

Extend the definition of $\kappa_1$ to $\kappa_1(\mathcal{G})$, analogously to $\kappa_0(\mathcal{G})$. Theorem 1(a) is mirrored by

**Theorem 2**

$$\rho_1(\mathcal{G}) \geq (2 - \kappa_1(\mathcal{G}))^{-1}.$$

This lower bound is generally stronger than that of Theorem 1(a) since $\kappa_1(G) \geq \kappa_0(G)$ always. But it is a disappointment in some ways (when compard with Theorem 1(b)) as it does not show that randomization via MRG always gives a significant improvement over the worst-case of Greedy. On the other hand, we have

**Theorem 3** *There is an absolute constant* $\epsilon \geq .0000025$ *such that*

$$\rho_1(\text{GRAPHS}) \geq .5 + \epsilon.$$

This theorem shows that randomization can strictly improve the worst-case performance of a greedy matching algorithm. In some ways, a *small triumph* for randomization. We do not of course believe that the lower bound of Theorem 3 is tight.

Our next result shows that the size of the matching produced by MRG is concentrated round its mean $\mu_1(G)$.

**Theorem 4** *Let $G$ be a graph with $m = m(G)$, $\mu_1 = \mu_1(G)$ and let $X = X(G)$ be the random size of the matching obtained by MRG in $G$. Then*

$$\Pr(\,|X - \mu_1| > \epsilon m\,) \le 2e^{-2\epsilon^2 m}$$

We next consider the performance of MRG on trees (and forests). We managed in [2] to establish $\rho_0(\text{FORESTS})$ by proving that Caterpillars have the worst RG performance. The situation here is more complicated and we have not established $\rho_1(\text{FORESTS})$ exactly. We have however managed to prove

**Theorem 5**

$$\rho_1(\text{FORESTS}) \ge \frac{18}{23} = .782608\ldots.$$

Thus the lower bound of MRG on FORESTS is strictly better than that of RG.

We finally consider the case of matching in uniform hypergraphs. Let $\mathcal{H} = (V, \mathcal{E})$ be an $r$-uniform hypergraph with $n = |V|$ and $N = |\mathcal{E}|$, i.e. for each *edge* $E_j \in \mathcal{E}$ $(j = 1, \ldots, N)$ we have $E_j \subseteq V$ and $|E_j| = r$. For each *vertex* $v \in V$, the *degree* of $d(v)$ of $v$ is the number of $E_j \in \mathcal{E}$ which contain $v$. The *average degree* $\bar{d} = \sum_{v \in V} d(v)/N$. A *matching* $M$ in $\mathcal{H}$ is a subset of $\mathcal{E}$ such that all $E_j \in M$ are mutually disjoint. We seek the maximum cardinality matching in $\mathcal{H}$. Maximum matching in graphs is the case $r = 2$ of this problem. Consider the following greedy algorithm for obtaining a "large" matching:

**HYPERGREEDY**

**begin**
  $X \leftarrow \emptyset$
  **while** $\mathcal{E} \neq \emptyset$ **do**
  **begin**
    **A:** Choose $E \in \mathcal{E}$
      $X \leftarrow X \cup \{E\}$;
      **for** all $E_j \in \mathcal{E}$ such that $E \cap E_j \neq \emptyset$ **do** $\mathcal{E} \leftarrow \mathcal{E} \setminus \{E_j\}$
  **end**
Output $X$
**end**

If $m(\mathcal{H})$ is the size of the maximum matching it follows [4] that in the worst case, HYPERGREEDY gives a matching of size $\lceil m/r \rceil$. We will assume that $E$ is chosen uniformly at random in $\mathcal{E}$ in Step **A**. Let $\mu(\mathcal{H})$ denote the expected size of the matching obtained by HYPERGREEDY and $\rho(\mathcal{H}) = \mu(\mathcal{H})/m(\mathcal{H})$. For any matching $M$ (including the empty matching) in a hypergraph $\mathcal{H}$, let us now define the *remainder* hypergraph $\mathcal{H} \setminus M$ having vertex set $V \setminus M = V \setminus \bigcup_{E_j \in M} E_j$ and edge set $\mathcal{E} \setminus M = \{E_j : E_j \subseteq V \setminus M\}$. Clearly $\mathcal{H} \setminus M$ is $r$-uniform if $\mathcal{H}$ is. Let us now define $\kappa(\mathcal{H})$ by

$$\kappa(\mathcal{H})^{-1} = \max_{M} \{\bar{d}(\mathcal{H} \setminus M) : M \text{ a matching in } \mathcal{H}\}.$$

**Theorem 6** *For $r$-uniform hypergraph $\mathcal{H}$,*

$$\rho(\mathcal{H}) \geq \frac{1}{r-(r-1)\kappa(\mathcal{H})}.$$

Thus Theorem 1(a) is the special case where $r = 2$.

## 2 Probability space for MRG

We now define a probability space over which we make our statements about MRG. Given $G = (V, E), |V| = n$, let $S(v)$ denote the set of orderings of $N(v) = N_G(v)$ so that $|S(v)| = |N(v)|!$ and let $S(V)$ denote the set of orderings of $V$. Let

$$\begin{aligned} \Omega = \Omega(G) \quad &= \quad \{(v_1, \pi_1, v_2, \pi_2, \ldots, v_n, \pi_n) : (v_1, v_2, \ldots, v_n) \in S(V) \\ &\qquad \text{and } \pi_i \in S(v_i) \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Thus to specify $\omega \in \Omega$ we order the vertices and then *independently* order the neighbourhood sets $N(v), v \in V$. We turn $\Omega$ into a probability space by making each $\omega \in \Omega$ equally likely. Given $\omega \in \Omega$ let $M = M(\omega)$ be the (greedy) matching obtained as follows:

**LIST-GREEDY**

**begin**

$M \leftarrow \emptyset, R \leftarrow V$;

**for** $i = 1, 2, \ldots n$ **do**

**begin**

if $v_i \in R$ and $N(v_i) \cap R \neq \emptyset$ **do**

**begin**

let $w$ be the first vertex of $N(v_i) \cap R$ (in the order $\pi_i$)

$M \to M \cup \{(v_i, w)\}$;

$R \to R \setminus \{v_i, w\}$

**end**;

**end**;

Output $M$

**end**

We claim that

$$M(\omega) \text{ has the same distribution as the matching chosen by MRG.} \qquad (2)$$

We prove this by induction on $|V|$. Some notation is helpful. For $\omega \in \Omega$ and $S \subseteq V$ let $\omega_S$ be obtained from $\omega$ by deleting $v, \pi(v)$ for $v \in S$ and all references to $S$ in the lists $\pi(x), x \notin S$. This may of course empty the adjacency list of a vertex $v \notin S$. It is convenient for the proof of Theorem 3 that such vertices remain on the list $\omega_S$.

**Lemma 1** *If $\omega$ is uniform over $\Omega(G)$ then $\omega_S$ is uniform over $\Omega(G \setminus S)$.*

**Proof** This follows from the fact that each $\omega' \in \Omega(G \setminus S)$ arises from the same number of $\omega \in \Omega(G)$ in this way. □

Returning to the comparison of $M(\omega)$ and the matching produced by MRG we observe that clearly the first edge $(v, w)$ of $M(\omega)$ has the same distribution as the first edge chosen by MRG. Putting $e = \{v, w\}$ we see, using Lemma 1, that by induction, the rest of the matching $M(\omega_e)$ has the same distribution as the rest of the matching produced by MRG.

Now let $X = X(\omega) = |M(\omega)|$ and write $\omega_v$ for $\omega_{\{v\}}$.

**Lemma 2** *If $v \in V$ then*

$$X(\omega) - 1 \leq X(\omega_v) \leq X(\omega).$$

**Proof** Let $K$ be the graph induced by the edges in the symmetric difference $M(\omega)\Delta M(\omega_v)$. If $M(\omega) \neq M(\omega_v)$ then one of the components of $K$ is an alternating path $P$ with $v$ as an endpoint. The lemma will follow from the fact that there can be no other component. To see this let $w$ be the first vertex of $V \setminus P$, in the ordering of $V$ defined by $\omega$ which has degree $d \geq 1$ in $K$. Let $C$ be the component of $K$ that contains $w$ and let $N_C(w)$ be the neighbour set of $w$ in $C$. Suppose that LIST-GREEDY is applied to $\omega$ and $\omega_v$. Because of the definition of $w$, no vertex in $N_C(w)$ is matched before $w$ is matched in both $\omega$ and $\omega_v$. Hence all vertices in $N_C(w)$ are available when LIST-GREEDY tries to match $w$ in either cases. This implies that $w$ is matched to the same vertex in both $\omega$ and $\omega_v$, contradicting the assumption that $w$ is in $K$.

□

**Corollary 1** **(a)** $\mu_1(G) - 1 \leq \mu_1(G \setminus \{v\}) \leq \mu_1(G)$, *for $v \in V$.*

**(b)** *If $v$ is left isolated by some maximum matching of $G$ then*

$$\rho_1(G \setminus \{v\}) \leq \rho_1(G).$$

**Proof** (a) This is a direct corollary of (2) and Lemmas 1 and 2.

(b) This follows from (a) and $m(G \setminus \{v\}) = m(G)$. □

## 3 Proof of Theorem 2

This is by induction on $|V|$. Let $\alpha(G) = (2 - \kappa_1(G))^{-1}$ and $\alpha(\mathcal{G}) = (2 - \kappa_1(\mathcal{G}))^{-1}$. If $|V| \leq 2$ then $r_1(G) = 1 = \alpha(G)$. In general it follows from Corollary 1(b) that we may assume $G$ has a perfect matching. So assume $|V| = n = 2m$ where $m = m(G)$. Then

$$\mu_1(G) = 1 + \frac{1}{2m} \sum_{u \in V} \frac{1}{d_u} \sum_{v \in N_G(u)} \mu_1(G \setminus \{u, v\}).$$

**Explanation:** $u$ denotes the randomly chosen vertex and $v$ its random neighbour. $1/2m$ is the probability we choose $u$ and then $1/d_u$ is the probability that we choose $v$. We add one edge to our matching and then $\mu(G \setminus \{u, v\})$ is the expected size of the matching produced after deleting $u$ and $v$.

However

$$m(G \setminus \{u, u\}) = \begin{cases} m - 1 & \text{if } (u, v) \text{ lies in some perfect matching} \\ m - 2 & \text{otherwise} \end{cases}$$

Hence, using the inductive hypothesis and putting $\alpha = \alpha(\mathcal{G})$,

$$\begin{aligned} \mu_1(G) &\geq 1 + \frac{1}{2m} \sum_{u \in V} \frac{\alpha}{d_u}((m-1) + (d_u - 1)(m-2)) \\ &= 1 + \alpha m - \frac{\alpha}{2m} \sum_{u \in V} \frac{2d_u - 1}{d_u} \\ &= \alpha m + 1 - \alpha \left( 2 - \frac{1}{n} \sum_{u \in V} \frac{1}{d_u} \right) \\ &\geq \alpha m, \end{aligned}$$

completing the induction. □

## 4 Proof of Theorem 3

We show that there exists $\epsilon \geq .0000025$ such that for all graphs $G$

$$r_1(G) \geq \tfrac{1}{2} + \epsilon. \tag{3}$$

Given $\epsilon$ suppose that there exists a graph $G = (V, E)$ which does not satisfy (3). We can assume by Corollary 1(b) that $G$ contains a perfect matching $M^*$, say, where $|M^*| = m$ and $|V| = n = 2m$. $G$ must also contain a *maximal* matching $\tilde{M}$ of size less than $(\frac{1}{2}+\epsilon)m$. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_\ell\}$ be a maximal set of vertex-disjoint alternating paths where each path $P_i$ contains two edges in $M^*$ and one edge in $\tilde{M}$. Now there is a set $E \subseteq M^*$ of $m - 2\ell$ edges not contained in any path $P \in \mathcal{P}$. Since $\mathcal{P}$ is maximal, no two edges in $E$ are connected by an edge in $\tilde{M}$. Also, since $\tilde{M}$ is maximal, at least one end-point of every edge in $E$ is also an end point of an edge in $\tilde{M}$. This shows that $|\tilde{M}| \geq (m - 2\ell) + \ell = m - \ell$. It follows that

$$\ell > (1 - 2\epsilon)n/4. \tag{4}$$

It will be helpful now to keep the nomenclature "choose" and "pick" as used in (1) i.e. MRG *chooses* a vertex at random and then *picks* a random neighbour. Let $u_t, v_t$ be the $t$'th vertices chosen and picked by MRG respectively, which happens at *time* $t$. If $u_t$ is an isolated vertex of $\Gamma$ at this time then we write $v_t = \star$ where $\star$ is a convenient symbol indicating this event.

Consider applying MRG to $G$ knowing that $G$ has the above structure. At any stage of such an execution a path $P \in \mathcal{P}$ is *complete* if none of its vertices has been chosen or picked up to now and *incomplete* otherwise. We introduce an indicator random variable $\delta(s), 1 \leq s \leq n$ for the event

$$\begin{aligned}\mathcal{E}(s) = \{&\text{(i) } u_s \text{ is the endpoint of a complete path } P = (u_s = x, y, z, w),\\ &\text{(ii) } v_s = y \text{ or } v_s \notin \{y, z\},\\ &\text{(iii) if } v_s \neq y \text{ then } y \in \{u_t, v_t\} \text{ for some } t > s,\\ &\text{(iv) if } v_s \neq y \text{ then } z \notin U_{t-1} \cup V_{t-1}\ \}\end{aligned}$$

where $U_i = U_i(s) = \{u_s, u_{s+1}, \ldots, u_i\}$ and $V_i = V_i(s) = \{v_s, v_{s+1}, \ldots, v_i\}$ for $i \geq s$.

Let

$$X = \sum_{s=1}^{n-1} \delta(s).$$

Let $M$ be the matching chosen by MRG. We prove next that

$$|M| \geq \frac{n}{4} + \frac{X}{4}. \tag{5}$$

To see this let $M_j, j = 0, 1, 2$ denote the set of $e \in M$ such that when MRG chooses $e$, the size of $E(\Gamma) \cap M^*$ decreases by $j$. Thus

$$\begin{aligned}|M| &= |M_0| + |M_1| + |M_2|,\\ |M^*| &= |M_1| + 2|M_2|,\end{aligned}$$

and so

$$\begin{aligned}|M| &= \tfrac{1}{2}|M^*| + \tfrac{1}{2}|M_1| + |M_0|\\ &\geq \frac{n}{4} + \frac{|M_0| + |M_1|}{2}.\end{aligned}$$

We prove (5) by showing

$$|M_0| + |M_1| \geq \frac{X}{2}. \tag{6}$$

So let $P, s, t, x, y, z, w$ be as in the definition of $\mathcal{E}(s)$, and let $\mathcal{E}(s)$ take place. If $v_s = y$ then $\{x, y\} \in M_1$. Otherwise either $\{y, v_t\}$ or $\{u_t, y\}$ is in $M_0 \cup M_1$.

Since $P$ is complete at the start of iteration $s$ it can only contribute once to $X$. At most two distinct values of $s$ can contribute to the same value of $t$. This proves (6).

Note that condition (iv) of the definition of $\mathcal{E}(s)$ was not used. Its use comes later when we have to lower bound the probability that $y$ has not become an isolated vertex before time $t$.

We must now bound $\Pr(\mathcal{E}(s))$ from below. So fix $s$ and condition on the graph $\Gamma$ at the end of iteration $s-1$. We write

$$\begin{aligned}
\mathcal{A}_1 &= \{u_s = x \text{ where } x \text{ is the endpoint of a complete path } x, y, z, w\} \\
\mathcal{A}_{2,1} &= \{v_s = y\}, \\
\mathcal{A}_{2,2} &= \{v_s \notin \{y, z\}\} \\
\mathcal{B} &= \{u_t = y \text{ and } z \notin U_{t-1} \cup V_{t-1} \text{ for some } t > s\}.
\end{aligned}$$

Since $\mathcal{A}_{2,1}$ and $\mathcal{A}_{2,2}$ are disjoint events, we have

$$\begin{aligned}
\Pr(\mathcal{E}(s)) &\geq \Pr(\mathcal{A}_1 \cap \mathcal{A}_{2,1}) + \Pr(\mathcal{A}_1 \cap \mathcal{A}_{2,2} \cap \mathcal{B}) \\
&= \Pr(\mathcal{A}_1)[\Pr(\mathcal{A}_{2,1}|\mathcal{A}_1) + \Pr(\mathcal{B}|\mathcal{A}_{2,2} \cap \mathcal{A}_1)\Pr(\mathcal{A}_{2,2}|\mathcal{A}_1)] \qquad (7)
\end{aligned}$$

Let $n_i$ denote the number of vertices of $\Gamma$ at the start of iteration $i$. Then since each iteration removes one or two vertices, it follows that

$$n - 2(i-1) \leq n_i \leq n - (i-1). \qquad (8)$$

Suppose that in the first $s-1$ iterations there are $s_j$ iterations in which $j$ vertices are deleted, $j = 1, 2$. Then,

$$\begin{aligned}
\Pr(\mathcal{A}_1) &\geq \frac{2\ell - 4s_2}{n - s_1 - 2s_2} \\
&\geq \frac{2\ell - 4(s-1)}{n - 2(s-1)} \qquad (9)
\end{aligned}$$

Next, let $d$ denote the current degree of the endpoint $x$ chosen by MRG, assuming $\mathcal{A}_1$. Then

$$\Pr(\mathcal{A}_{2,1} \mid \mathcal{A}_1) = \frac{1}{d}, \tag{10}$$

and

$$\Pr(\mathcal{A}_{2,2} \mid \mathcal{A}_1) \geq (1 - 2/d)^+, \tag{11}$$

where $\xi^+ = \max\{0, \xi\}$. Hence (7) implies

$$\Pr(\mathcal{E}(s)) \geq \Pr(\mathcal{A}_1)\left[\Pr(\mathcal{B}|\mathcal{A}_{2,2} \cap \mathcal{A}_1) + \frac{2}{d}\left(\frac{1}{2} - \Pr(\mathcal{B}|\mathcal{A}_{2,2} \cap \mathcal{A}_1)\right)\right].$$

Since MRG chooses a vertex randomly, we have

$$\begin{aligned} \Pr(\mathcal{B}|\mathcal{A}_{2,2} \cap \mathcal{A}_1) &\leq \Pr(u_t = y \text{ and } z \notin U_{t-1} \text{ for some } t > s|\mathcal{A}_{2,2} \cap \mathcal{A}_1) \\ &\leq 1/2. \end{aligned}$$

It follows that

$$\Pr(\mathcal{E}(s)) \geq \Pr(\mathcal{A}_1) \Pr(\mathcal{B}|\mathcal{A}_{2,2} \cap \mathcal{A}_1). \tag{12}$$

Unless stated otherwise, we use Pr to denote the probability conditional on $\mathcal{A}_1 \cap \mathcal{A}_{2,2}$ for the rest of this section. Our main task is to find a lower bound for $\Pr(\mathcal{B})$, the probability of $\mathcal{B}$ conditional on $\mathcal{A}_1 \cap \mathcal{A}_{2,2}$.

Assume the occurrence of $\mathcal{A}_1 \cap \mathcal{A}_{2,2}$, and for $t$ satisfying $s+1 \leq t \leq n$, define the events,

$$\mathcal{C}(t) = \{y, z \notin U_{t-1} \cup V_{t-1}\}, \qquad \mathcal{B}(t) = \mathcal{C}(t) \cap \{u_t = y\}.$$

Note that $\mathcal{B}(t), s+1 \leq t \leq n$, are disjoint events and that

$$\Pr(\mathcal{B}) = \sum_{t=s+1}^{n} \Pr(\mathcal{B}(t)). \tag{13}$$

Now $u_t$ is a random choice from the $n_t$ available vertices and $y$ is available when $\mathcal{C}(t)$ occurs. Thus we have

$$\Pr(\mathcal{B}(t)) = \mathbf{E}\left(\frac{1}{n_t}\Pr(\mathcal{C}(t))\right) \geq \frac{1}{n}\Pr(\mathcal{C}(t)). \tag{14}$$

It thus follows from (12) that

$$\mathbf{E}(\delta(s)) \geq \Pr(\mathcal{A}_1)\left(\frac{1}{n}\sum_{t=s+1}^{n}(1-\Pr(\{y,z\}\cap U_{t-1}\neq\emptyset)-\Pr(\{y,z\}\cap V_{t-1}\neq\emptyset))\right). \tag{15}$$

We next estimate $\Pr(\bar{\mathcal{C}}(t))$. Now

$$\Pr(\{y,z\}\cap U_{t-1}\neq\emptyset) \leq \mathbf{E}\left(\sum_{i=s+1}^{t-1}\frac{2}{n_{i-1}}\right) \leq \frac{2(t-s)}{n-2s}, \tag{16}$$

but $\Pr(\{y,z\}\cap V_{t-1}\neq\emptyset)$ is more difficult to estimate since it depends on the structure of $\Gamma$. We will explicitly compute an upper bound for $\Pr(z\in V_{t-1})$ and then we can double this to obtain an upper bound for $\Pr(\{y,z\}\cap V_{t-1}\neq\emptyset)$.

Suppose now that $(u_s,v_s)=(x,x')$ where $x'\notin\{y,z\}$. We have to estimate $\Pr(z\in V_{t-1})$ when MRG is run on the graph $\Gamma\setminus\{x,x'\}$, or equivalently when LIST-GREEDY is run on $\omega_e$, $e=\{x,x'\}$ and $\omega$ is chosen randomly from $\Omega(\Gamma)$. We will find it useful to couple the process with LIST-GREEDY run on $\omega$ itself. (Here $x$ is not necessarily the first vertex in $\omega$'s list.)

Let $\Gamma_{s+1}=\Gamma\setminus\{x,x'\}$, $\Gamma_{s+2}$, $\Gamma_{s+3}$, …denote the sequence of subgraphs produced by vertex deletion as LIST-GREEDY runs on $\omega_e$. Similarly, let $\Gamma=\Gamma^r_{s+1}$, $\Gamma^r_{s+2}$, $\Gamma^r_{s+3}$, … denote the corresponding sequence of subgraphs as LIST-GREEDY runs on $\omega$. Suppose also that the unconditioned process (input $\omega$) chooses $u^r_{s+1}$, $u^r_{s+2}$, $u^r_{s+3}$, …and picks $v^r_{s+1}$, $v^r_{s+2}$, $v^r_{s+3}$, …. Let $U^r_t = \{u^r_{s+1}, u^r_{s+2}, \ldots, u^r_t\}$, $V^r_t = \{v^r_{s+1}, v^r_{s+2}, \ldots, v^r_t\}$ (with $U^r_s = V^r_s = \emptyset$) and

$$\Delta_t = (U_t \cup V_t) \bigtriangleup (U^r_t \cup V^r_t).$$

When the following event $\mathcal{F}_{t-1}$ occurs, the relationship between $\Pr(z \in V_{t-1})$ and $\Pr(z \in V^r_{t-1})$ is relatively simple to analyze. Let

$$\mathcal{F}_i = \{u^r_{j+1} \notin \Delta_j : s \leq j \leq i\}, \qquad s \leq i \leq t-1,$$

and note that $\mathcal{F}_i$ decreases with $i$.

**Lemma 3** *Suppose $\mathcal{F}_{t-1}$ occurs. Then for $i$ satisfying $s \leq i \leq t-1$*

*(i)* $|\Delta_i| \leq 2$,

*(ii)* $\Gamma_{i+1}$ *is an induced subgraph of* $\Gamma^r_{i+1}$,

**Proof** We use induction on $i$. The base case $i = s$ is clearly trivial. So assume (i) and (ii) are true for some $i > s$ and that $u_{i+1} = u^r_{i+1} \in V(\Gamma_{i+1})$ (using $\mathcal{F}_{t-1}$). Note that induction hypothesis (ii) implies that $V(\Gamma^r_{i+1}) = V(\Gamma_i) \cup \Delta_i$. Now if $v^r_{i+1} = \star$, then it follows from the induction hypothesis $\Gamma_{i+1} \subseteq \Gamma^r_{i+1}$ that $v_{i+1} = v^r_{i+1} = \star$, in which case both $\Gamma_{i+1}$ and $\Gamma^r_{i+1}$ lose the same vertex $u^r_{i+1} = u_{i+1}$ and the induction goes through. On the other hand, if $v^r_{i+1} \neq \star$ then there are two cases:

(A) If $v^r_{i+1} \in \Delta_i$, then $\Delta_i$ loses $v^r_{i+1}$ (but may gain $v_{i+1}$ depending on whether $v_{i+1} = \star$), and induction goes through;

(B) If $v^r_{i+1} \notin \Delta_i$, then $v_{i+1} = v^r_{i+1}$ (using induction hypothesis (ii)), and induction goes through in this case too. □

We next bound $\Pr(z \in V_{t-1})$ from above.

$$\begin{aligned}
\Pr(z \in V_{t-1}) &\leq \Pr(\{z \in V_{t-1}\} \cap \mathcal{F}_{t-1}) + \Pr(\bar{\mathcal{F}}_{t-1}) \\
&= \sum_{i=s+1}^{t-1} \Pr(\{z = v_i\} \cap \mathcal{F}_{t-1}) + \Pr(\bar{\mathcal{F}}_{t-1}) \\
&\leq \sum_{i=s+1}^{t-1} \Pr(\{z = v_i\} \cap \mathcal{F}_{i}) + \Pr(\bar{\mathcal{F}}_{t-1}) \\
&= \sum_{i=s+1}^{t-1} \Pr(z = v_i \mid \{z \in V(\Gamma_i)\} \cap \mathcal{F}_i) \Pr(\{z \in V(\Gamma_i)\} \cap \mathcal{F}_i). \\
&\qquad + \Pr(\bar{\mathcal{F}}_{t-1}) \qquad (17)
\end{aligned}$$

To estimate the right hand side of (17), we prove

**Lemma 4** *(i)* $\Pr(\bar{\mathcal{F}}_{t-1}) \leq \sum_{i=s+1}^{t-1} \frac{2}{n-2i+2}$,

*(ii)* $\Pr(\{z \in V(\Gamma_i)\} \cap \mathcal{F}_i) \leq \Pr(\{z \in V(\Gamma_i^r)\} \cap \mathcal{F}_i), \qquad s \leq i \leq t-1,$

*(iii)* $\Pr(z = v_i \mid \{z \in V(\Gamma_i)\} \cap \mathcal{F}_i) \leq \left(3 + \frac{6}{n_{i-1}}\right) \Pr(z = v_i^r \mid \{z \in V(\Gamma_i^r)\} \cap \mathcal{F}_i), \qquad s \leq i \leq t-1.$

**Proof** (i)

$$\begin{aligned}
\Pr(\bar{\mathcal{F}}_{t-1}) &\leq \sum_{i=s+1}^{t-1} \Pr(u_i^r \in \Delta_i \mid \mathcal{F}_{i-1}) \\
&\leq \sum_{i=s+1}^{t-1} \frac{2}{n-2i+2},
\end{aligned}$$

since $|\Delta_i| \leq 2, n_i \geq n - 2(i-1)$ and $u_i^r$ is chosen randomly from $V(\Gamma_i^r)$.

(ii) This follows immediately from the fact that the occurrence of $\mathcal{F}_i$ implies that $\Gamma_i \subseteq \Gamma_i^r$ (Lemma 3(ii)).

(iii) Conditional on the additional events that $\Gamma_i = K$ and $\Gamma_i^r = L$ where $z \in V(K) \subseteq V(L)$ and $|V(L) \setminus V(K)| \leq 2$, and writing $h = |V(K)| = n_{i-1}$ and $\tilde{\Pr}$ for the new conditional probability, we have

$$\begin{aligned}
\tilde{\Pr}(z = v_i) &= \frac{1}{h} \sum_{v \in N_K(z)} \frac{1}{d_K(v)} \\
&\leq \frac{1}{h} \sum_{v \in N_K(z)} \frac{3}{d_L(v)} \\
&\leq \frac{3}{h} \sum_{v \in N_L(z)} \frac{1}{d_L(v)} \\
&\leq \frac{3(h+2)}{h} \tilde{\Pr}(z = v_i^r).
\end{aligned}$$

□

Applying the lemma to the RHS of (17) we get

$$\begin{aligned}
\Pr(z \in V_{t-1}) &\leq \left(3 + \frac{6}{n_s}\right) \sum_{i=s+1}^{t-1} \Pr(\{z = v_i^r\} \cap \mathcal{F}_i) + \sum_{i=s+1}^{t-1} \frac{2}{n - 2i + 2} \\
&\leq \left(3 + \frac{6}{n_s}\right) \sum_{i=s+1}^{t-1} \Pr(z = v_i^r) + \sum_{i=s+1}^{t-1} \frac{2}{n - 2i + 2} \\
&\leq \left(3 + \frac{6}{n_s}\right) \Pr(z \in V_{t-1}^r) + \sum_{i=s+1}^{t-1} \frac{2}{n - 2i + 2}. \qquad (18)
\end{aligned}$$

Now $\Pr(z \in V_{t-1}^r)$ is easy to estimate, since the given $\omega$ is coupled with $\omega_e$ where $x, y, z, w$ is a random path. (This is most easily seen if we consider that we run LIST-GREEDY on $\omega$, choose a random path $x, y, z, w$ and then run LIST-GREEDY on $\omega_e$). Thus $z$ is chosen randomly from at least $2\ell - 4(s-1)$ vertices and

$$\Pr(z \in V_{t-1}^r) \leq \frac{t - s}{2\ell - 4(s-1)}.$$

Combining this with (18) gives

$$\begin{aligned}
\Pr(z \in V_{t-1}) &\leq \left(3 + \frac{6}{n_s}\right) \frac{t-s}{2\ell - 4(s-1)} + \sum_{i=s+1}^{t-1} \frac{2}{n - 2i + 2} \\
&\leq \left(3 + \frac{6}{n_s}\right) \frac{t-s}{2\ell - 4(s-1)} + \frac{2(t-s)}{n - 2t}. \qquad (19)
\end{aligned}$$

We can now finish the proof of Theorem 3 by using (5) and (15) together with (4), (9), (16) and (19). We will not try for the best possible value of $\epsilon$ obtainable from these inequalities (suffice it to say that careful estimations and computer computation can give a slightly larger bound). Instead we plump for something smaller but simpler.

So let us consider only $s \leq \alpha n, t \leq \beta n, \alpha \leq \beta$ for some small values of $\alpha, \beta$. Then we obtain for $n$ large,

$$\begin{aligned}
\Pr(\mathcal{A}_1) &\geq \frac{1 - 2\epsilon - 8\alpha}{2 - 4\alpha}, \\
\Pr(\{y, z\} \cap U_{t-1} \neq \emptyset \mid \mathcal{A}_1 \cap \mathcal{A}_{2,2}) &\leq \frac{2\beta}{1 - 2\alpha}, \qquad (20) \\
\Pr(\{y, z\} \cap V_{t-1} \neq \emptyset \mid \mathcal{A}_1 \cap \mathcal{A}_{2,2}) &\leq \frac{16\beta}{1 - 2\epsilon - 8\alpha} + \frac{4\beta}{1 - 2\beta}, \qquad (21)
\end{aligned}$$

where Pr here denotes the unconditioned probability law. Now increasing $\beta$ only increases the RHS's of (20) and (21) and so we take $\alpha = \beta$. Hence from (15) (with the summation only taken to $\alpha n$),

$$\frac{1}{n}\mathbf{E}(X) \geq \frac{\alpha^2}{2} \frac{1 - 2\epsilon - 8\alpha}{2 - 4\alpha} \left(1 - \frac{6\alpha}{1 - 2\alpha} - \frac{16\alpha}{1 - 2\epsilon - 8\alpha}\right). \qquad (22)$$

The theorem is now proved by choosing $\alpha, \epsilon$ so that the RHS of (22) is at least $\epsilon$, since this contradicts $r_1(G) < \frac{1}{2} + \epsilon$. A simple calculation shows that this holds for $\alpha = .01$ and $\epsilon = .00001$. □

## 5 Concentration near the mean

We now prove Theorem 4. The proof is essentially identical to the proof of Theorem 3 of [2]. We give it here for completeness. Let $Y_i$, $(i = 0, 1, \ldots, m)$ be the Doob martingale induced by the first $i$ selections (choice plus pick) of MRG on $G$, *i.e.* $Y_i = \mathbf{E}(X \mid \text{first } i \text{ choices})$. Clearly $Y_i = K + \mu_1(H)$ for some integer $K \leq i$ and subgraph $H$ of $G$. In fact $K = i$ unless $H = \emptyset$. Also

$$\begin{aligned} Y_{i+1} &= K + 1 + \mu_1(H \setminus \{u, v\} && \text{if } H \text{ contains an edge,} \\ &= K && \text{otherwise,} \end{aligned}$$

where $uv$ is the $i + 1$'th choice of edge. Thus,

$$\begin{aligned} Y_{i+1} - Y_i &= 1 + \mu_1(H \setminus \{u, v\} - \mu_1(H)) && \text{if } H \text{ contains an edge,} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Thus if $H$ contains an edge,

$$\begin{aligned} Y_{i+1} - Y_i &= 1 + \mu_1(H \setminus \{u, v\} - \mu_1(H)) \\ &\leq 1, \qquad \text{since } \mu_1(H \setminus \{u, v\}) \leq \mu_1(H) \end{aligned}$$

Furthermore,

$$\begin{aligned} Y_{i+1} - Y_i &\geq \mu_1(H \setminus \{u\} - \mu_1(H)), \\ & \qquad\qquad \text{since } \mu_1(H \setminus \{u, v\}) \geq \mu_1(H \setminus \{u\}) - 1 \\ &\geq -1, \qquad\qquad \text{since } \mu_1(H \setminus \{u\}) \geq \mu_1(H) - 1, \end{aligned}$$

where all inequalities follow from Lemma 2.

Thus $|Y_{i+1} - Y_i| \leq 1$ whether or not $H$ has an edge. Hence $\{Y_i\}$ is a bounded difference martingale sequence, and it follows from the Hoeffding-Azuma inequality (see Bollobás [1], McDiarmid [5]) that

$$\Pr(|X-\mu| > \epsilon m) \le 2e^{-2(\epsilon m)^2/m} = 2e^{-2\epsilon^2 m}$$

□

**Corollary 2** *If $\{G_m\}$ is a graph sequence such that $m(G_m)(=m) \to \infty$, and $\omega_m \to \infty$ (arbitrarily slowly), then*

$$\Pr(\mu_1(G_m) - \omega_m\sqrt{m} \le X(G_m) \le \mu(G_m) + \omega_m\sqrt{m}) \to 1$$

**Proof** Put $\epsilon = \omega_m/\sqrt{m}$ in Theorem 4. □

# 6 Trees

We prove that for any tree $T$,

$$\mu_1(T) \ge \frac{18}{23}m(T) + \frac{4}{23}. \tag{23}$$

It is only necessary to prove this for trees with a perfect matching. Let $T=(V,E)$ be a tree that has a perfect matching of size $m$. The proof of (23) will be by induction on $m$. Assume inductively that the theorem holds for all trees with maximum matchings of size $m-1$ or less. For the moment let us be general and try to prove that $\mu_1(T) \ge \alpha m + \beta$ for $m \ge 1$. In the proof we will have to place various restrictions on $\alpha$ and $\beta$. $\alpha = \frac{18}{23}$ and $\beta = \frac{4}{23}$ will turn out to be the values that maximize $\alpha$.

All trees with a perfect matching and $m \le 6$ have been checked (by computer) and have been found to satisfy (23). Also one can easily derive a recurrence relation for $\sigma_m = \mu_1(\mathrm{P}_m)$ where $\mathrm{P}_m$ is a path of length $m$. From the solution to this equation we can deduce that $\sigma_m \approx .876681m$ and that (23) is satisfied for paths. We can therefore assume from now on that $T$ is a tree with at least 14 vertices and is not a path.

Let $M$ be the set of edges in the perfect matching. Note that the perfect matching of any tree is unique (indeed the edge set of two distinct perfect matchings would contain an alternating cycle). Let $v^*$ be the unique matching neighbour of $v$. Let $L$ be the set of vertices of degree one, $L'$ be the set of vertices neighboring $L$ and $K = V \setminus (L \cup L')$. Let $\ell = |L| = |L'|$ (since $T$ has a perfect matching) and $k = |K|$. Then let $\mathcal{F}(F)$ denote the set of trees in a forest $F$.

Let

$$f(v) = \begin{cases} 0 & v \in L', \\ 1 & v \notin L'. \end{cases}$$

Then

$$\begin{aligned}
\mu_1(T) &= 1 + \frac{1}{2m} \sum_{v \in V} \frac{1}{d(v)} \sum_{w \in N(v)} \mu_1(T \setminus \{v, w\}) \\
&= 1 + \frac{1}{2m} \sum_{v \in V} \frac{1}{d(v)} (\mu_1(T \setminus \{v, v^*\}) + \sum_{(v,w) \notin M} \mu_1(T \setminus \{v, w\})) \\
&\geq 1 + \frac{1}{2m} \sum_{v \in V} \frac{1}{d(v)} \left( \sum_{\hat{T} \in \mathcal{F}(T \setminus \{v, v^*\})} (\alpha m(\hat{T}) + \beta) + \sum_{(v,w) \notin M} \sum_{\hat{T} \in \mathcal{F}(T \setminus \{v,w\})} (\alpha m(\hat{T}) + \beta) \right) \\
&= 1 + \frac{1}{2m} \sum_{v \in V} \frac{1}{d(v)} \Bigg( \alpha(m-1) + \beta(d(v) + d(v^*) - 2) + \\
&\qquad \sum_{(v,w) \notin M} (\alpha(m-2) + \beta(d(v) + d(w) + f(v) + f(w) - 4)) \Bigg) ).
\end{aligned}$$

Let us collect terms. First of all those involving $\alpha$ are

$$\begin{aligned}\frac{1}{2m}\sum_{v\in V}\frac{1}{d(v)}\left(m-1+\sum_{(v,w)\notin M}(m-2)\right) &= \frac{1}{2m}\sum_{v\in V}\frac{1}{d(v)}(m-1+(m-2)(d(v)-1)) \\ &= \frac{1}{2m}\sum_{v\in V}\frac{1}{d(v)}(1+(m-2)d(v)) \\ &= m-2+\frac{1}{2m}\sum_{v\in V}\frac{1}{d(v)}.\end{aligned}$$

For $\beta$ we have the terms

$$\frac{1}{2m}\sum_{v\in V}\frac{1}{d(v)}\left(d(v)+d(v^*)-2+\sum_{(v,w)\notin M}(d(v)+d(w)+f(v)+f(w)-4)\right)$$

$$= \frac{1}{2m}\left(\sum_{(v,w)\in E}\left(\frac{d(w)}{d(v)}+\frac{d(v)}{d(w)}\right)+\sum_{v\in V}\frac{1}{d(v)}\sum_{(v,w)\notin M}(f(v)+f(w))\right)+$$

$$\frac{1}{2m}\sum_{v\in V}\frac{1}{d(v)}\left(d(v)-2+\sum_{(v,w)\notin M}(d(v)-4)\right)$$

$$= \frac{1}{2m}\left(\sum_{(v,w)\in E}\left(\frac{d(w)}{d(v)}+\frac{d(v)}{d(w)}\right)+\sum_{v\in V}\frac{1}{d(v)}\sum_{(v,w)\notin M}(f(v)+f(w))\right)$$

$$+\frac{1}{2m}\sum_{v\in V}\frac{1}{d(v)}(d(v)^2-4d(v)+2)$$

$$= \frac{1}{2m}\left(\sum_{(v,w)\in E}\left(\frac{d(w)}{d(v)}+\frac{d(v)}{d(w)}\right)+\sum_{v\in V}\frac{1}{d(v)}\sum_{(v,w)\notin M}(f(v)+f(w))\right)$$

$$+\frac{1}{m}\sum_{v\in V}\frac{1}{d(v)}-\left(2+\frac{1}{m}\right),$$

since $\sum_{v\in V} d(v) = 4m-2$.

Employing these expressions we obtain

$$\begin{aligned}\mu_1(T) &\geq 1+\alpha(m-2)-\left(2+\frac{1}{m}\right)\beta+\frac{\alpha+2\beta}{2m}\sum_{v\in V}\frac{1}{d(v)}+\\ &\quad \frac{\beta}{2m}\left(\sum_{(v,w)\in E}\left(\frac{d(w)}{d(v)}+\frac{d(v)}{d(w)}\right)+\sum_{v\in V}\frac{1}{d(v)}\sum_{(v,w)\notin M}(f(v)+f(w))\right)\end{aligned}$$

Now as $T$ is not an isolated edge we have

$$\frac{d(w)}{d(v)}+\frac{d(v)}{d(w)} \geq \begin{cases} 2 & \text{if } v\notin L \\ 2\frac{1}{2} & \text{if } v\in L \end{cases}$$

and so

$$\sum_{(v,w)\in E}\left(\frac{d(w)}{d(v)}+\frac{d(v)}{d(w)}\right) \geq 4m-2+\frac{\ell}{2}.$$

Then observe that

$$\sum_{v\in V}\sum_{(v,w)\notin M}\frac{f(v)}{d(v)} = \sum_{v\in K}\frac{d(v)-1}{d(v)}$$

and

$$\sum_{v\in V}\sum_{(v,w)\notin M}\frac{f(w)}{d(v)} = \sum_{v\in K}\frac{d(v)-1}{d(v)}+\sum_{(v,w)\in L'\times K}\left(\frac{1}{d(v)}-\frac{1}{d(w)}\right).$$

Hence

$$\begin{aligned}\mu_1(T) &\geq 1+\alpha(m-2)-\left(2+\frac{1}{m}\right)\beta+\frac{\alpha}{2m}\sum_{v\in V}\frac{1}{d(v)}+\\ &\quad +\frac{\beta}{2m}\left(4m-2+\frac{\ell}{2}+2+|K|\sum_{v\notin K}\frac{1}{d(v)}+\sum_{(v,w)\in L'\times K}\left(\frac{1}{d(v)}-\frac{1}{d(w)}\right)\right)\end{aligned}$$

$$= \quad \alpha m + \beta + \left(1 - 2\alpha + \beta - \frac{2\beta}{m}\right) + \frac{\alpha}{2m}\sum_{v\in V}\frac{1}{d(v)} + \frac{\beta}{2m}\left(-\frac{3\ell}{2} + \sum_{v\in L'}\frac{2}{d(v)} + \sum_{(v,w)\in L'\times K}\left(\frac{1}{d(v)} - \frac{1}{d(w)}\right)\right). \tag{24}$$

We prove next that

$$\sum_{v\in L'}\frac{2}{d(v)} + \sum_{(v,w)\in L'\times K}\left(\frac{1}{d(v)} - \frac{1}{d(w)}\right) \geq \frac{\ell}{2} + 1. \tag{25}$$

If $v \in L'$ we let $a(v)$ denote the number of neighbours of $v$ in $L'$ and $b(v)$ denote the number of neighbours of $v$ in $K$. Since $d(w) \geq 2$ for $w \in K$ we may prove (25) by showing

$$\sum_{v\in L'}\frac{b(v)+2}{a(v)+b(v)+1} - \frac{1}{2}\sum_{v\in L'} b(v) \geq \frac{\ell}{2} + 1. \tag{26}$$

This will be shown to be true for all trees with the Property $\mathcal{P}$: no two vertices of degree one share a common neighbour (a tree with a perfect matching certainly has property $\mathcal{P}$). We proceed by induction on $\ell$. Let $\Delta$ denote the LHS of (26).

**Base Cases:** $\ell = 2, T$ is a path of length $m \neq 2$ and $L' = \{x, y\}$.

**(i)** $m = 1$: $a(x) = a(y) = 1, b(x) = b(y) = 0$ and $\Delta = 2$.

**(ii)** $m > 2$: $a(x) = a(y) = 0, b(x) = b(y) = 1$ and $\Delta = 2$.

Now consider the general case and $\ell \geq 3$. $\mathcal{P}$ implies that $L'$ contains a vertex $x$ of degree 2 with one neighbour $z \in L$ and one neighbour $w \notin L$. (Root $T$ and let $z \in L$ be at maximum depth. Its neighbour $x$ has the required property). Suppose first that $w \in L'$. Remove $x, z$ to obtain $T'$ and let $\Delta'$ be the associated value of $\Delta$. Note that this and all subsequent deletions that we consider preserve Property $\mathcal{P}$, as required for the induction. Then

$$\begin{aligned}\Delta &= 1+\frac{b(w)+2}{a(w)+b(w)+1}-\frac{b(w)+2}{a(w)+b(w)}+\Delta' \\ &\geq \frac{1}{2}+\Delta' \\ &\geq \frac{1}{2}+\frac{1}{2}(\ell-1)+1,\end{aligned}$$

by induction. The second inequality uses $a(w) \geq 1$ and $a(w)+b(w) \geq 2$ from $\ell \geq 3$.

If $w \in K$ let $x_1, x_2, \ldots x_d$ denote $w$'s other neighbours. If $d \geq 2$ then we obtain $T'$ by removing $x, z$. Then

$$\begin{aligned}\Delta &= \frac{3}{2}-\frac{1}{2}+\Delta' \\ &> \frac{\ell}{2}+1.\end{aligned}$$

If $d = 1$ then there is a path $(y_0 = z, y_1 = x, y_2 = w, y_3 = x_1, \ldots, y_p)$ where $y_i \in K$ for $i \geq 2$ and $y_2, y_3, \ldots, y_{p-1}$ have degree 2 and $y_p$ has degree at least 3. We obtain $T'$ by deleting $y_0, y_1, \ldots, y_{p-1}$. If $y_p \notin L'$ then $\Delta = 1 + \Delta'$ as in the previous case. If $y_p \in L'$ then

$$\begin{aligned}\Delta &= 1+\frac{b(y_p)+2}{a(y_p)+b(y_p)+1}-\frac{b(y_p)+1}{a(y_p)+b(y_p)}+\Delta' \\ &\geq 1+\Delta' \\ &> \frac{\ell}{2}+1.\end{aligned}$$

This completes the proof of (26) and hence (25). Substituting (25) into (24) gives

$$\mu_1(G) \geq \alpha m + \beta + (1-2\alpha+\beta) + \frac{\alpha}{2m}\sum_{v \in V}\frac{1}{d(v)} - \frac{\beta}{2m}(\ell+3) \qquad (27)$$

Now by convexity

$$\sum_{v \notin L} \frac{1}{d(v)} \geq \frac{a}{2} + \frac{b}{3}$$

where

$$\begin{aligned} a + b &= 2m - \ell \\ 2a + 3b &= \textstyle\sum_{v \notin L} d(v) \quad = \quad 4m - 2 - \ell. \end{aligned}$$

The solution to this set of equations is $a = 2m + 2 - 2\ell$ and $b = \ell - 2$. This gives

$$\begin{aligned} \sum_{v \in V} \frac{1}{d(v)} &\geq \ell + \frac{1}{2}(2m + 2 - 2\ell) + \frac{1}{3}(\ell - 2) \\ &= m + \frac{\ell}{3} + \frac{1}{3}. \end{aligned}$$

Substituting into (27) gives

$$\mu_1(G) \geq \alpha m + \beta + \left(1 - \frac{3\alpha}{2} + \beta\right) + \frac{l}{6m}(\alpha - 3\beta) + \frac{1}{6m}(\alpha - 9\beta).$$

Since $T$ is not a path we have $\ell \geq 3$. We also have $m \geq 7$. Assume that $\alpha - 3\beta \geq 0$, $2\alpha - 9\beta \leq 0$ and $1 - \frac{59\alpha}{42} + \frac{4\beta}{7} \geq 0$. Then

$$\begin{aligned} \mu_1(G) &\geq \alpha m + \beta + \left(1 - \frac{3\alpha}{2} + \beta\right) + \frac{1}{3m}(2\alpha - 9\beta) \\ &\geq \alpha m + \beta + \left(1 - \frac{59\alpha}{42} + \frac{4\beta}{7}\right) \\ &\geq \alpha m + \beta. \end{aligned}$$

Three restrictions have been placed on $\alpha$ and $\beta$. These are

$$3\beta \leq \alpha \leq 9\beta/2 \text{ and } \frac{59}{42}\alpha - \frac{4\beta}{7} \leq 1.$$

Maximizing $\alpha$ over this set of inequalities yields $\alpha = \frac{18}{23} = .782608\cdots$ and $\beta = \frac{4}{23} = .1739\cdots$. This proves Theorem 5.

We have not been able to compute $\rho_1(\textsc{Forests})$ exactly, in contradistinction to Theorem 1(c). At one stage we thought that paths would be the worst-case trees. However, a path with 12 vertices does not minimise $r_1$ over trees with 12 vertices, instead one takes a path with 10 vertices and then adds two new leaves attached to the two middle vertices. This graph has a value of $\rho_1 = .832844\cdots$. This seems to make the exact computation of $\rho_1$ harder than that for $\rho_0$.

# 7 Hypergraphs

We now prove Theorem 6. We use induction on $m(\mathcal{H})$. Note that a remainder hypergraph of any $\mathcal{H} \setminus M$ is also a remainder hypergraph of $\mathcal{H}$. As basis, if $m(\mathcal{H}) = 1$ it is clear that $\rho(\mathcal{H}) = 1$ and $\kappa(\mathcal{H}) \leq 1$, so the Theorem holds. Otherwise, let us fix a maximum matching $M^*$ of cardinality $m \geq 2$. This will cover $rm$ vertices and thus $n - rm$ will lie outside the matching edges. Let $\mathcal{N}_i$ denote the set of edges in $\mathcal{H}$ which have $i$ vertices outside $M^*$ and $N_i = |\mathcal{N}_i|$, for $i = 0, 1, \ldots, r-1$. Clearly $N = m + \sum_{i=0}^{r-1} N_i$. Let us denote the right side in the inequality of the Theorem by $\alpha$. Now, by induction,

$$\begin{aligned} \mu(\mathcal{H}) &\geq 1 + \frac{1}{N}\left(m\alpha(m-1) + \sum_{i=0}^{r-1} N_i \alpha(m - r + i)\right) \\ &= \alpha m + 1 - \frac{\alpha}{N}\left(m + (N-m)r - \sum_{i=0}^{r-1} iN_i\right). \end{aligned}$$

**Explanation:** the randomly chosen edge is either (i) in $M^*$ (with probability $m/N$) and we can expect to get at least $\alpha(m-1)$ from the remaining edges, or (ii) in $\mathcal{N}_i$ (probability $N_i/N$) and we can expect at least $\alpha(m-r+1)$ from the remaining edges.

Thus the induction will succeed if

$$\alpha^{-1} \geq r - \frac{m(r-1) + \sum_{i=0}^{r-1} iN_i}{N}.$$

Now each vertex outside $M$ is in at least one $E_j$, so $\sum_{i=0}^{r-1} iN_i \geq n - rm$. Thus it suffices to have

$$\begin{aligned} \alpha^{-1} &\geq r - \frac{m(r-1) + (n-rm)}{N} \\ &= r - \frac{n-m}{N}, \end{aligned}$$

which is true if

$$\alpha^{-1} \geq r - \frac{n(r-1)}{rN},$$

since $m \leq n/r$. But $\bar{d}(\mathcal{H}) = rN/n$, so it suffices that

$$\alpha^{-1} \geq r - \frac{(r-1)}{\bar{d}(\mathcal{H})},$$

which follows from $\alpha^{-1} = r - (r-1)\kappa$. □

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