Perfect matchings in random graphs with prescribed minimal degree

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Abstract

We consider the existence of perfect matchings in random graphs with n vertices (or $n+n$ vertices in the bipartite case) and m random edges, *subject to a lower bound on minimum vertex degree.* A random bipartite graph without isolated vertices and $m > n$ edges with high probability (**whp**) has a perfect matching iff the average vertex degree is $0.5 \log n + \log\log n + c_n$, $c_n \to \infty$ however slow. A random graph with minimum degree at least two **whp** has a matching that matches all the vertices except "odd-man-out" vertices, one per each isolated cycle of odd length, and one for the remaining vertex set if its cardinality is odd. So, for n even, **whp** the random graph has a perfect matching iff it does not have isolated odd cycles.

1 Introduction

To quote from Lovász [19], "the problem of the existence of 1-factors (perfect matchings), the solution of which (the Kőnig-Hall theorem for bipartite graphs and Tutte's theorem for the general case) is an outstanding result making this probably the most developed field of graph theory". Erdős and Rényi ([9], [10]) found a way to use these results for a surprisingly sharp study of existence of perfect matchings in random graphs. For $B_{n,m}$, a random bipartite graph with $n+n$ vertices and $m = n(\ln n + c_n)$ random edges, they proved [9] that

$$\begin{aligned}\lim_{n\to\infty} \mathbf{Pr}(B_{n,m} \text{ has a perfect matching}) &= \lim_{n\to\infty} \mathbf{Pr}(\delta(B_{n,m}) \geq 1) \\ &= \begin{cases} 0 & c_n \to -\infty, \\ e^{-2e^{-c}}, & c_n \to c, \\ 1 & c_n \to \infty, \end{cases}\end{aligned}$$

where δ denotes minimum degree. Of course, minimum degree at least one is a trivial necessary condition for the existence of a perfect matching. The Hall theorem turned out to be perfectly tailored for use in combination with probabilistic techniques, pioneered in [9] several years earlier. Even though Tutte's theorem for the non-bipartite case is considerably more involved, in [10] Erdős and Rényi managed to extend the analysis to the random graph $G_{n,m}$, a random general

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graph with n vertices and $m = \frac{n}{2}(\ln n + c_n)$ edges, showing that

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \mathbf{Pr}(G_{n,m} \text{ has a perfect matching}) = \lim_{n\to\infty} \mathbf{Pr}(\delta(G_{n,m}) \geq 1) = \begin{cases} 0 & c_n \to -\infty, \\ e^{-e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases}$$

In both cases a perfect matching becomes likely as soon as one has sufficiently many random edges for the minimum degree to be at least one with high probability (**whp**). This has led researchers to consider the existence of perfect matchings in models of a random graph in which the minimum degree requirement is always satisfied. Perhaps the first result along these lines is due to Walkup [25]. He considered a κ-out model $B_{\kappa-\text{out}}$ of a random bipartite graph, again with $n + n$ vertices $V_1 + V_2$. Here each vertex $v \in V_i$ "chooses" κ random neighbours in its complementary class V_{3-i}. Walkup showed that

$$\lim_{n\to\infty} \mathbf{Pr}(B_{\kappa-out} \text{ has a perfect matching}) = \begin{cases} 0 & \kappa = 1 \\ 1 & \kappa \geq 2 \end{cases}$$

Frieze [11] proved a non-bipartite version of this result, the argument being based on Tutte's theorem and considerably harder. Very recently Karoński and Pittel [15] have proven **whp** the existence of a perfect matching in what they called the $B_{(1+e^{-1})-out}$ graph, a subgraph of B_{2-out}, obtained from B_{1-out} by letting each of its degree 1 vertices select another random neighbor in the complementary class. Observe that in all of these results [25], [11] and [15] the number of random edges depends linearly on the number of vertices, and the minimum degree has been raised to 2, in a sharp contrast with the case m being of order $n \log n$. Here is why. When there are order $n \ln n$ random edges, there are few vertices of degree 1 and they are far apart. In sparser models, with minimum degree 1, **whp** there will be a linear (in n) number of vertices of degree 1, and some two vertices of degree 1 will have a common neighbor, which rules out a perfect matching. In the case of random regular graphs it turns out that minimum degree 3 is required, Bollobás [3]: Let G_r denote a random r-regular graph on vertex set $[n]$, n even. Then

$$\lim_{n\to\infty} \mathbf{Pr}(G_r \text{ has a perfect matching}) = \begin{cases} 0 & r = 2, \\ 1 & r = 1 \text{ or } r \geq 3. \end{cases}$$

The case $r = 1$ is trivial since then G_r is itself a perfect matching of $[n]$. G_2 is **whp** a collection of $O(\ln n)$ disjoint cycles and they will all have to be even for G_2 to have a perfect matching. The meat of the result is therefore in the case $r \geq 3$ and this follows from r-connectivity and Tutte's theorem.

Another approach was considered by Bollobás and Frieze [6]. Let $\mathcal{G}_{n,m}^{\delta\geq\kappa}$ denote the set of graphs with vertex set $[n]$, m edges and minimum degree at least κ. Let $G_{n,m}^{\delta\geq\kappa}$ be sampled uniformly from $\mathcal{G}_{n,m}^{\delta\geq\kappa}$. By conditioning on minimum degree 1, say, we will need 50% fewer random edges to get a perfect matching **whp**: Let $m = \frac{n}{4}(\ln n + 2\ln\ln n + c_n)$.

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \mathbf{Pr}(G_{n,m}^{\delta\geq 1} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to -\infty \textit{ sufficiently slowly}, \\ e^{-\frac{1}{8}e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases} \tag{1}$$

The restriction "sufficiently slowly" may seem out of place, but bear in mind that if n is even and $m = n/2$ then the probability of a perfect matching is 1. The precise threshold between $n/2$ and $\frac{1}{4}n \ln n$ for the non-existence of a perfect matching was not determined. Using the approach developed in the present paper for the bipartite case, we have found that "sufficiently slowly" in (1) can be replaced simply by "and $m > n/2$". (For $m = n/2 + 1$, say, the likely graph, with minimum degree 1 at least, consists of $n/2 - 3$ isolated edges, and two paths, each consisting of 3 vertices.) The study in [6] was extended in Bollobás, Fenner and Frieze [4] who considered the probability that $G^{\delta\geq\kappa}_{n,m}$ has $\lfloor\kappa/2\rfloor$ disjoint Hamilton cycles plus a further disjoint perfect matching if κ is odd.

In the present paper we continue this line of research. We first consider the bipartite version of (1). Let $\mathcal{B}^{\delta\geq\kappa}_{n,m}$ denote the set of bipartite graphs with vertex set $[n]+[n]$, m edges and minimum degree at least κ. Let $B^{\delta\geq\kappa}_{n,m}$ be sampled uniformly from $\mathcal{B}^{\delta\geq\kappa}_{n,m}$.

Theorem 1. *Let* $m = \frac{n}{2}(\ln n + 2\ln\ln n + c_n)$. *Then*

$$\lim_{n\to\infty} \mathbf{Pr}(B^{\delta\geq 1}_{n,m} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to -\infty,\ m > n, \\ e^{-\frac{1}{4}e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases} \tag{2}$$

(As in the case of $\mathcal{G}^{\delta\geq 1}_{n,m}$, we observe that the threshold for m is reduced by the factor of 2, compared to that of the random graph $B_{n,m}$.) The RHS expression in (2) is the limiting probability that no two vertices of degree 1 have a common neighbor. Thus, the probability that a perfect matching exists is (close to) 1 when either $m = n/2$ or c_n is large, and the probability is very small for m everywhere in between, except c_n not far to the left from 0.

The next natural question is: How many random edges are needed if we constrain the minimum degree to be at least 2, so ruling out the possibility of two vertices of degree 1 having a common neighbour?

In this paper we only consider the non-bipartite graphs. To cover both even and odd values of the number of vertices, it is convenient, and natural, to say that a graph $G = (V, E)$ has a perfect matching if $\mu^*(G) = \lfloor |V|/2 \rfloor$, where $\mu^*(G)$ is the maximum matching number of G.

Unlike the bipartite case, with a positive limiting probability the "sparse" graph $G^{\delta\geq 2}_{n,cn}$ may have (short) isolated odd cycles. This observation rules out a "**whp**-type" result for probability of a perfect matching. Let $X(G)$ stand for the total number of odd isolated cycles in G. Clearly

$$\mu^*(G) \leq \nu(G) := \left\lfloor \frac{|V| - X(G)}{2} \right\rfloor.$$

Let $\mu^*_n = \mu^*(G^{\delta\geq 2}_{n,cn})$, $X_n = X(G^{\delta\geq 2}_{n,cn})$ and $\nu_n = \mu(G^{\delta\geq 2}_{n,cn})$.

Theorem 2. *Let* $\liminf c > 1$. *Then*

$$\lim_{n\to\infty} \mathbf{Pr}(\mu^*_n = \nu_n) = 1,$$

and X_n is, in the limit, Poisson (λ),

$$\lambda = \lambda_n := \frac{1}{4}\log\frac{1+\sigma}{1-\sigma} - \frac{\sigma}{2}, \quad \sigma := \frac{\rho}{e^\rho - 1},$$

and ρ satisfies

$$\frac{\rho(e^\rho - 1)}{e^\rho - 1 - \rho} = 2c.$$

In particular,

$$\lim_{n\to\infty} \mathbf{Pr}(G^{\delta\geq 2}_{n,cn} \text{ has a perfect matching}) = \begin{cases} e^{-\lambda}, & \text{if } n \text{ even}, \\ e^{-\lambda} + \lambda e^{-\lambda}, & \text{if } n \text{ odd}. \end{cases} \tag{3}$$

Thus the subgraph obtained by deletion of isolated odd cycles **whp** has a perfect matching The RHS in (3) is the limiting probability that the total number of isolated odd cycles is 0 (n even), or 1 (n odd). Notice that $c = 1$ corresponds to the random 2-regular (non-bipartite) graph, which typically has $\Theta(\log n)$ isolated cycles, both odd and even. Sure enough, the explicit term in the RHS of (3) approaches zero as $c \downarrow 1$, since $\lambda \to \infty$.

Theorem 2 does leave open the case where the number of edges $m = 2 + o(n)$ and so it is not quite as tight as Theorem 1.

Here is an interesting application of Theorem 2. Consider the Erdős-Rényi random graph $G(n, m)$, $m = cn$, for $\liminf c > 1/2$, i.e. the supercritical phase. By consecutive deletion of the vertices of degree 1 at most, we obtain a 2-core, the largest subgraph of $G(n, m)$ with minimum degree 2 at most. Let ν, μ stand for the number of vertices and the number of edges in the 2-core. Conditioned on ν, μ, and the vertex set, the 2-core of $G(n, m)$ is distributed as $G^{\delta\geq 2}_{\nu,\mu}$. Since **whp** μ, ν are of order n, and μ/ν is bounded away from 1, we see that **whp** the 2-core of the giant component of $G(n, m)$ has a perfect matching.

Among other things, the proof of Theorem 2 is based on an asymptotic analysis [2] of a matching algorithm initially discovered and studied by Karp and Sipser [16]. A related analysis for the bipartite graph $B^{\delta\geq 2}_{n,m}$ is considerably more than a technical extension of that in [2], basically because of some serious complications due to bipartiteness. It is shown in [12] that $B^{\delta\geq 2}_{n,m}$ has a perfect matching **whp**when $m = cn$, $c \geq 2$ constant.

To conclude our discussion, for integer $k \geq 1$, let graph G have property $\mathcal{A}_k$ if G contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. Bollobás, Cooper, Fenner and Frieze [5] show that for $k \geq 2$, there exists a constant $c_k \leq 2(k+2)^3$ such that if $c \geq c_k$, $G^{\delta\geq k+1}_{n,cn}$ has property $\mathcal{A}_k$. Thus the current paper deals with the property $\mathcal{A}_1$ and proves a sharp result. It is reasonable to conjecture that the true value for c_k is $(k + 1)/2$. Note that if $c = (k + 1)/2$ and cn is integer then $G^{\delta\geq k+1}_{n,cn}$ is a random $(k + 1)$-regular graph and this is known to have the property $\mathcal{A}_k$ **whp**, Robinson and Wormald [24], Kim and Wormald [17].

2 Enumerating some bipartite graphs.

In our probabilistic model, the sample space $\mathcal{B}^{\delta\geq k}_{n,m}$ is the set of all bipartite graphs on the bipartition $[n] + [n]$ with m edges, and the minimum degree at least k. The probability measure is uniform, i.e. each sample graph $B^{\delta\geq k}_{n,m}$ is assigned the same probability, $N_k(n, m)^{-1}$, where $N_k(n, m) = |\mathcal{B}^{\delta\geq k}_{n,m}|$. We will obtain a sharp asymptotic formula for $N_k(n, m)$, as a special case for the number of bipartite graphs meeting more general conditions on vertex degrees.

Let the ν_1-tuple $\mathbf{c} = (c_1, \ldots, c_{\nu_1})$ and the ν_2-tuple $\mathbf{d} = (d_1, \ldots, d_{\nu_2})$ of nonnegative integers be given. Introduce $N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu)$, $\boldsymbol{\nu} = (\nu_1, \nu_2)$, the total number of bipartite graphs with μ edges, such that $a_i \geq c_i$, $(i \in [\nu_1])$, and $b_j \geq d_j$, $(j \in [\nu_2])$. Of course, $N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) = 0$ if $\mu < \sum_i c_i$, or $\mu < \sum_j d_j$. So we assume that $\mu \geq \max\{\sum_i c_i, \sum_j d_j\}$

Define

$$G_{\mathbf{c}}(x) = \prod_{i\in[\nu_1]} f_{c_i}(x); \tag{4}$$

$$H_{\mathbf{d}}(y) = \prod_{j\in[\nu_2]} f_{d_j}(y), \tag{5}$$

where

$$f_t(z) \quad = \quad \sum_{\ell \ge t} \frac{z^\ell}{\ell!} = e^z - \sum_{\ell < t} \frac{z^\ell}{\ell!}. \tag{6}$$

The following estimates will be proved in Appendix A, along with some other lengthy computations.

Lemma 1. *Suppose that $\nu_1, \nu_2, \mu \to \infty$ are such that that $\nu_1, \nu_2 = O(\mu)$ and $\mu = O(\nu_i \log \nu_i)$, $i = 1, 2$. Let $G_{\mathbf{c}}(x)$ and $H_{\mathbf{d}}(y)$ be defined by (4) and (5).*

(i) *Suppose that $\mu^{-1} \le_b r_1, r_2 = O(\log \mu)$. Then*

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) \quad \sim \quad \mu! \frac{G_{\mathbf{c}}(r_1) H_{\mathbf{d}}(r_2)}{(r_1 r_2)^\mu} \left[e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})} \mathbf{Pr}(R = \mu)\mathbf{Pr}(S = \mu) + O(e^{-\Omega(\log^5 \mu)}) \right], \tag{7}$$

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) \quad \le_b \quad \mu! \frac{G_{\mathbf{c}}(r_1) H_{\mathbf{d}}(r_2)}{(r_1 r_2)^\mu \sqrt{\nu_1 \nu_2 r_1 r_2}} e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}, \tag{8}$$

the last estimate holding without the condition $\nu_1, \nu_2, \mu \to \infty$, where $Y_i = Po(r_1; \ge c_i)$, $Z_j = Po(r_2; \ge d_j)$ are all independent, and $R = \sum_i Y_i$, $S = \sum_j Z_j$.

(ii) *Suppose also that $\max_i c_i = O(1)$, $\max_j d_j = O(1)$, and $\mu > \max\{\sum_i c_i, \sum_j d_j\}$. Then there exist (unique) positive roots ρ_1, ρ_2 of (100) and (101), and*

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) \sim \mu! \frac{G_{\mathbf{c}}(\rho_1) H_{\mathbf{d}}(\rho_2)}{(\rho_1 \rho_2)^\mu} e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})} \cdot \mathbf{Pr}(R = \mu)\mathbf{Pr}(S = \mu), \tag{9}$$

where $Y_i = Po(\rho_1; \ge c_i)$, $Z_j = Po(\rho_2; \ge c_j)$.

Furthermore

$$\mathbf{Pr}(R = \mu) \sim \frac{1}{(2\pi \sum_i \mathbf{Var}(Y_i))^{1/2}}, \quad \text{or} \quad \mathbf{Pr}(R = \mu) \sim e^{-\sigma_1} \frac{\sigma_1^{\sigma_1}}{\sigma_1!},$$

dependent upon whether $\sigma_1 := \mu - \sum_i c_i$ approaches infinity or stays bounded, with the analogous formula for $\mathbf{Pr}(S = \mu)$.

Corollary 3. *Suppose $n = O(m)$, $kn < m = O(n \log n)$. Then*

$$N_k(n, m) \sim m! \left(\frac{f_k(\rho)^n \mathbf{Pr}\left(\sum_i Y_i = m\right)}{\rho^m} \right)^2 \exp\left(-\frac{n^2}{2m^2} \mathbf{E}^2[(Y)_2] \right), \tag{10}$$

where Y is Poisson $(\rho; \ge k)$ such that $\mathbf{E}Y = r = m/n$. Note that

$$\mathbf{E}[(Y)_2] = \begin{cases} r^2, & k = 0, \\ \rho r, & k = 1, \\ \rho r/(1 - e^{-\rho}), & k = 2. \end{cases} . \tag{11}$$

Further

$$\mathbf{Pr}\left(\sum_i Y_i = m \right) \sim (2\pi n \mathbf{Var} Y)^{-1/2}, \qquad \text{if } m - kn \to \infty, \tag{12}$$

and

$$\mathbf{Pr}\left(\sum_i Y_i = m \right) \sim e^{-\sigma} \frac{\sigma^\sigma}{\sigma!} \qquad \text{if } m - kn > 0 \text{ is fixed.} \tag{13}$$

As we will see, these results are all we need to evaluate (bound) the probabilities arising in the proofs of Theorem 1 and Theorem 2. We will also need a crude upper bound for the fraction of bipartite graphs in question, with the maximum degree exceeding m^{α}. This bound is already implicit in the preceding analysis! Indeed, from (90), (92), (93), and the observation that the factor

$$\mathbf{Pr}^2(\sum_i Y_i = m)\exp\left(-\frac{n^2}{2m^2}\mathbf{E}^2[(Y)_2]\right) \tag{14}$$

in (10) is $\exp(-\Theta(\log^2 n))$, it follows that, for $\alpha' < \alpha < 1/3$, this fraction is $e^{-m^{\alpha'}}$ at most.

One is tempted to call this "overpowering both the conditioning and the fudge factor". Needless to say, this trick would work for the counts (fractions) of other graph classes, as long as the degrees restrictions are so severe that the probability that Y_i, Z_j meet them is negligible compared to the factor in (14).

3 Proof of Theorem 1

We will use Hall's necessary and sufficient condition for the existence of a perfect matching in a bipartite graph to prove (1).

The random graph $B^{\delta\ge 1}_{n,m}$ has no perfect matching iff for some $k \ge 2$ there exists a k-*witness*. A k-witness is a pair of sets $K \subseteq R, L \subseteq C$, or $K \subseteq C, L \subseteq R$, such that $|K| = k, |L| = k-1$ and $N(K) \subseteq L$. Here $N(K)$ denotes the set of neighbours of vertices in K. A k-witness is *minimal* if there does not exist $K' \subset K, L' \subset L$ such that (K', L') is a k'-witness, where $k' < k$. It is straightforward that if (K, L) is a minimal k-witness then every member of L has degree at least two in $B(K \cup L)$, the subgraph of $B^{\delta\ge 1}_{n,m}$ induced by $K \cup L$. Therefore $B(K \cup L)$ has at least $2(k-1)$ edges. We can restrict our attention to $k \le n/2$ since for $k > n/2$ we can consider $\tilde{C} = C \setminus L, \tilde{R} = R \setminus K$. For $2 \le k \le n/2$, let $W_{n,k,\mu}$ denote the random number of minimal k-witnesses, such that $B(K \cup L)$ has μ edges, $\mu \ge 2(k-1)$. Actually, since $k \le n/2$, we also have $\mu \le m - n$.

(i) Suppose $m = O(n \log n)$ and $m \ge (1/3 + \epsilon) n \log n$, $\epsilon > 0$. Let us prove that **whp** $B^{\delta\ge 1}_{n,m}$ has no k-witnesses with $k \ge 3$, i.e.

$$\mathbf{Pr}\left(\sum_{k\ge 3, \mu \ge 2(k-1)} W_{n,k,\mu} = 0\right) \to 1, \quad n \to \infty.$$

It suffices to show that

$$\sum_{k\ge 3, \mu\ge 2(k-1)} E_{n,k,\mu} \to 0, \quad E_{n,k,\mu} := \mathbf{E} W_{n,k,\mu}. \tag{15}$$

Let us bound $E_{n,k,\mu}$. For certainty, suppose that $K \subset R$, $L \subset C$. We can choose a pair (K, L) in $\binom{n}{k}\binom{n}{k-1}$ ways. (K, L) being a witness imposes the above listed conditions on degrees of the subgraph induced by $K \cup L$. If we delete the row set K, we get a remainder graph, which is a bipartite graph with bipartition (R', C), $R' = R \setminus K$; it has $m - \mu$ edges and every vertex in $R' \cup (C \setminus L)$ has degree 1 at least. We bound N_1, the total number of those subgraphs, and N_2 the total number of the remainder graphs using Lemma 1 (i), emphasizing the possibility to choose the corresponding parameters r_1, r_2 anyway we want. The product of these two bounds divided by the asymptotic expression for $N_1(n, m)$ in Corollary 3 provides an upper bound for the probability that (K, L) is a k-witness with μ edges. Multiplying this bound by $2\binom{n}{k}\binom{n}{k-1}$, we get a bound for $E_{n,k,\mu}$. To implement this program, we consider separately $k \le m^\beta$ and $k \ge m^\beta$, where $\beta \in (0,1)$ will be specified in the course of the argument.

Let $k \le m^\beta$. Pick $\alpha' < \alpha = (1-\beta)/2$. From the note following Corollary 3, with probability $1 - e^{-m^{\alpha'}}$ at least, the maximum vertex degree in the uniformly random bipartite graph is m^α at most. So, backpedaling a bit, we will consider $\mu \le m^\gamma$, ($\gamma := (1+\beta)/2$), only. To bound N_1 we use (85) with $r_1 = \mu/k, r_2 = \mu/(k-1)$, and to bound N_2 we use (8) with $r_1 = r_2 = \rho$. Here ρ is the parameter of Y_i in Corollary 3, the root of $xf_0(x)/f_1(x) = r$, $r := m/n$, so that

$$\rho = r(1 - e^{-\rho}) < r, \quad \rho = r - \Theta(re^{-r}). \tag{16}$$

The r_i for N_1 seem natural, if one interprets them as parameters of Poissons approximating the vertex degrees that should add to μ on either side of the subgraph induced by $K \cup L$. Since k, μ are relatively small, $r_1 = r_2 = \rho$ should be expected to deliver a good enough bound for N_2. Most importantly, this choice does the job!

After cancellations and trivial tinkering, the resulting bound is

$$\begin{aligned} E_{n,k,\mu} \quad \le_b \quad & \frac{m\binom{n}{k}\binom{n}{k-1}}{\mu(m-\mu)\binom{m}{\mu}} \cdot \rho^{2\mu} \left(\frac{k}{\mu}\right)^\mu \left(\frac{k-1}{\mu}\right)^\mu \\ \times \quad & \frac{f_1(\mu/k)^k f_1(\mu/(k-1))^{k-1} f_0(\rho)^{k-1}}{f_1(\rho)^{2k-1}} \\ \times \quad & \frac{\exp\left(-\frac{1}{2}\frac{(n-k)\mathbf{E}(Y)_2}{m-\mu} \cdot \frac{(k-1)\rho^2 + (n-k+1)\mathbf{E}(Y)_2}{m-\mu}\right)}{\exp\left(-\frac{1}{2}\frac{(n\mathbf{E}(Y)_2)^2}{m^2}\right)}. \end{aligned} \tag{17}$$

Some explanation: $k-1$ vertices from L in the remaining graph have degrees not bounded away from zero, whence the factor $f_0(\rho)^{k-1} = e^{\rho(k-1)}$ in the second line, and $k-1$ usual Poissons (ρ), each with the second factorial moment equal ρ^2, contributing $(k-1)\rho^2$ in the last line. Also, we have used $f_1(\mu/(k-1))^{k-1}$ where we could have used the smaller $f_2(\mu/(k-1))^{k-1}$.

The last line fraction is of order $O(1)$, as $\mathbf{E}(Y)_2 = \Theta(\rho^2)$. Further, since $\log f_1(z) = \log(e^z - 1)$ is concave,

$$\begin{aligned} & k \log f_1\left(\frac{\mu}{k}\right) + (k-1)\log f_1\left(\frac{\mu}{k-1}\right) - (2k-1)\log f_1(\rho) \le \\ & \qquad (2k-1)\left(\log f_1\left(\frac{2\mu}{2k-1}\right) - \log f_1(\rho)\right) \le (2k-1)(\log f_1)'(\rho)\left(\frac{2\mu}{2k-1} - \rho\right) \\ & \qquad \le 2\mu - (2k-1)\rho + 3\mu e^{-\rho}. \end{aligned} \tag{18}$$

Using the last observations and $\mu! = \Theta(\mu^{1/2}(\mu/e)^\mu)$, we see that $E_{n,k,\mu}$ is of order $E^*_{n,k,\mu}$ at most, where

$$E^*_{n,k,\mu} = \frac{n^{2k-1}e^{-k\rho}}{k!(k-1)!} \cdot \frac{(m-\mu)!}{m!\mu!} k^{2\mu}\rho^{2\mu} \cdot \exp(3\mu e^{-\rho}). \tag{19}$$

First, since $2(k-1) \le \mu \le m^\gamma$,

$$\frac{E^*_{n,k,\mu+1}}{E^*_{n,k,\mu}} = \frac{k^2\rho^2}{(m-\mu)(\mu+1)} = O(m^{-1/3}\log^2 n),$$

so that

$$\sum_{2(k-1) \le \mu \le m^\gamma} E^*_{n,k,\mu} \sim E^*_{n,k,2(k-1)}.$$

Second

$$\begin{aligned} \frac{E^*_{n,k+1,2k}}{E^*_{n,k,2(k-1)}} &= \frac{n^2(k+1)^{4k}\rho^4 e^{-\rho}}{k(k+1)(2k-1)2k(m-2k+1)(m-2k+2)k^{4(k-1)}} \\ &\le_b \frac{n^2}{m^2}\rho^4 e^{-\rho} = O(\rho^2 e^{-\rho}). \end{aligned}$$

Therefore

$$\sum_{3\le k\le m^\gamma} E^*_{n,k,2(k-1)} \sim E^*_{n34} \le_b \frac{n^5}{m^4}\rho^4 e^{-3\rho} \sim n e^{-3m/n} = O(n^{-3\epsilon}),$$

as $m \ge (1/3+\epsilon)n\log n$. In summary,

$$\sum_{\substack{3\le k\le m^\gamma\\ 2(k-1)\le\mu\le m^\gamma}} E_{n,k,\mu} = O(n^{-3\epsilon}). \tag{20}$$

Consider now $k \ge m^\beta$. This time we use (7) not only for N_1 but for N_2 as well, using for the latter $r_1 = (\rho n-\mu)/(n-k)$ and $r_2 = (\rho n - \mu)/n$. That the latter r_i are positive follows from $\mu \le m - n/2$ and (16). (For μ, k not being relatively small anymore, the count N_2 of the remaining graphs would hardly be well bounded via the previous choice $r_1 = r_2 = \rho \sim m/n$. What we have chosen turns out to be a working compromise between that old choice and the "naive" $r_1 = (m-\mu)/(n-k)$, $r_2 = (m-\mu)/n$.) The resulting bound is

$$\begin{aligned} E_{n,k,\mu} &\le_p \exp\left(\frac{(n\mathbf{E}(Y)_2)^2}{2m^2}\right) E'_{n,k,\mu}, \\ E'_{n,k,\mu} &= \rho^{2m}\frac{\binom{n}{k}^2}{\binom{m}{\mu}}\cdot\left(\frac{k}{\mu}\right)^{2\mu}\left(\frac{n-k}{\rho n-\mu}\right)^{m-\mu}\left(\frac{n}{\rho n-\mu}\right)^{m-\mu} \\ &\times \frac{f_1(\frac{\mu}{k})^k f_1(\frac{\mu}{k-1})^{k-1} f_1(\frac{\rho n-\mu}{n-k})^{n-k} f_1(\frac{\rho n-\mu}{n})^{n-k+1} f_0(\frac{\rho n-\mu}{n})^{k-1}}{f_1(\rho)^{2n}}. \end{aligned} \tag{21}$$

(We use the notation $a_n \le_p b_n$ to indicate that a_n/b_n is polynomially large, at most.) Using again convexity of $\log f_1(z)$ and denoting $h = (k-1)(\rho n-\mu)/n$, we obtain that the logarithm of the last line fraction is less than

$$2n\log f_1\left(\rho - \frac{h}{2n}\right) - 2n\log f_1(\rho) + h \le -h((\log f_1)'(\rho) - 1) = -\frac{h}{e^\rho - 1} \le 0.$$

Thus the fraction is bounded, 1 at most, like its counterpart for $k \le m^\beta$. (Our search for the proper r_1, r_2 was driven, in fact, by desire to make that fraction bounded again!)

Introduce $x = k/n$, $y = \mu/n$. Using the Stirling formula for factorials, we obtain easily then that

$$E'_{n,k,\mu} \le_p \exp(nH(x,y)),$$

where

$$H(u,v) = 2r\log\rho + 2H(u) - rH(v/r) + 2v\log u/v + (r-v)\log\frac{1-u}{(\rho-v)^2},$$

$(u \in (x_n, 1/2]$, $v \in (0,\rho))$, $x_n := \frac{m^\beta}{n}$, and

$$H(w) = w\log 1/w + (1-w)\log 1/(1-w).$$

It follows that

$$\begin{aligned} H_v(u,v) &= \log\frac{u^2(\rho-v)^2}{v(r-v)(1-u)} - 2\frac{\rho-r}{\rho-v}, \\ H_{vv}(u,v) &= \frac{1}{r-v} - \frac{2}{\rho-v} - \frac{1}{v} - \frac{2(\rho-r)}{(\rho-v)^2} < 0. \end{aligned}$$

So $H_v(u,v)$ decreases with v, and $H_v(u,0+)=\infty$, $H_v(u,\rho-)=-\infty$. Hence, given u, $H(u,v)$ attains its maximum at a unique root $v(u)$ of the equation

$$\frac{(\rho-v)^2u^2}{(r-v)(1-u)v} = \exp\left(\frac{2(\rho-r)}{\rho-v}\right). \tag{22}$$

By (16), $\rho<r$ and $\rho-r=O(re^{-r})\to 0$; so we should expect $v(u)$ to be close to $v^*(u)$, the root of (22) with ρ replaced by r, i.e.

$$v^*(u) = \frac{u^2}{1-u+u^2}r \le \frac{r}{3},$$

$(u\le 1/2)$. Careful computations reveal that

$$H^*(u,v^*(u)) = r\log(1-u+u^2)+2H(u),$$

where $H^*(u,v)$ is obtained from $H(u,v)$ by replacing ρ with r. Furthermore, as the RHS of (22) is $1+O(e^{-r})$, it can be shown that

$$v(u) = v^*(u)(1+O(e^{-r})).$$

In this setting, strictly speaking, $v(u)$ is also a function of ρ, and so is $H(u,v(u))$, both explicitly and implicitly, via $v(u)$. Since $H_v(u,v(u))=0$, the derivative of $H(u,v(u))$ with respect to ρ is just the partial derivative, which is

$$\frac{2r}{\rho} - \frac{2(r-v(u))}{\rho-v(u)} = \frac{2v(u)(\rho-r)}{\rho(\rho-v(u))} = O(e^{-r});$$

therefore

$$H(u,v(u)) = H^*(u,v^*(u)) + O(e^{-r}(r-\rho)) = H^*(u,v^*(u)) + O(re^{-2r}).$$

Using

$$\log(1-u+u^2)\le -u/2,\ (u\le 1/2),\quad (1-u)\log(1-u)^{-1}\le u,$$

we see that, for $u\in(m^\beta/n,1/2]$ (and $m\ge(1/3+\epsilon)n\log n$),

$$\begin{aligned} H^*(u,v^*(u)) &\le -u(r/2-2-2\log(1/u)) \\ &\le -u((1/6+\epsilon/2)\log n - 2(1-\beta)\log n + O(\log\log n)) \\ &\le -cu\log n, \end{aligned}$$

where $c=c(\beta)>0$ if $\beta>11/12-\epsilon/4$. So, for this choice of β,

$$\begin{aligned} H(u,v(u)) &= H^*(u,v^*(u)) + O(r^{-2r}) \\ &\le -cm^\beta n^{-1}\log n + O(e^{-2m/n}\log n) \\ &\le -m^\beta n^{-1}. \end{aligned}$$

This inequality shows that

$$E'_{n,k,\mu} \le_p \exp(-m^\beta) \Rightarrow E_{n,k,\mu} \le \exp(-0.5m^\beta),$$

as the fudge factor in (21) is only $\exp(O(\log^2 n))$. Consequently

$$\sum_{\substack{m^\beta \le k \le n/2 \\ \mu \le m}} E_{n,k,\mu} \le \exp(-0.4m^\beta). \tag{23}$$

Combining (20) and (23) we obtain

$$\sum_{\substack{3 \le k \le m^\beta \\ 2(k-1) \le \mu \le m^\gamma}} E_{n,k,\mu} + \sum_{\substack{m^{1/3} < k \le n/2 \\ \mu \le m}} E_{n,k,\mu} = O(n^{-3\epsilon}),$$

so that

$$\mathbf{Pr}\left(\sum_{k \ge 3} \sum_{\mu \ge 2(k-1)} W_{n,k,\mu} = 0 \right) = 1 - O(n^{-3\epsilon}). \tag{24}$$

(ii) Turn now to the 2-witnesses. From (19), it follows that

$$E_{n,2,1} \le_b \frac{n^3 e^{-2\rho} \rho^4}{m^2} = O(n \log^2 n e^{-2m/n}) = O(e^{-c_n}), \tag{25}$$

with c_n defined by the notation

$$m = \frac{n}{2}(\log n + 2 \log\log n + c_n).$$

Case 1 $c_n \to \infty$.
Assume first that $m \le 2n \log n$. Then (25) shows that, with probability more than $1 - e^{-c_n} \to 1$, there are no 2-witnesses. By (25), with probability $1 - O(n^{-1/2})$ at least, there are no 3-witnesses either. Thus, with probability approaching 1, there exists a perfect matching.

If $m > 2n \log n$ then **whp** $\delta(B_{n,m}) \ge 1$ and $B_{n,m}$ has a perfect matching. The result in this case follows immediately.

Case 2: $c_n \to c \in (-\infty, \infty)$.
We want to prove that $W_{n,2,1}$, the number of 2-witnesses, is, in the limit, Poisson $(e^{-c}/4)$. We do so via the factorial moments method. To evaluate $\mathbf{E}(W_{n,2,1})$ sharply, we notice that in this case $N_1 = 2\binom{n}{2}n$ exactly, and for N_2 we use the part (ii) of Lemma 1 with $r_1 = r_2 = \rho$. So (compare to (17)

$$E_{n,2,1} \sim \frac{2n\binom{n}{2}}{m^2} \cdot \rho^4 e^{-2\rho} \sim \frac{1}{4} e^{-c_n} \to \lambda := \frac{1}{4} e^{-c}. \tag{26}$$

We need to show that, for each $t \ge 2$

$$\lim_{n \to \infty} \mathbf{E}(W_{n,2,1})_t = \lambda^t, \quad \lambda = \frac{1}{4} e^{-c}.$$

To simplify our task, let us consider instead $W^*_{n,2,1}$, the total number of vertex-disjoint 2-witnesses. The difference $W_{n,2,1} - W^*_{n,2,1}$ is **whp** (24) $\mathcal{W}_n$ at most, where $\mathcal{W}_n$ is the total number of subset pairs (K, L), $K \subset R$, $L \subset C$, or $K \subset C$, $L \subset R$, such that $|K| = 3$, $|L| = 1$, and $L = N(K)$. Analogously to (17),

$$\mathbf{E}\mathcal{W}_n \le_b \frac{n\binom{n}{3}\rho^6}{\binom{m}{3}} e^{-3\rho} = O(n^{-1/2}).$$

Therefore, $W_{n,2,1} = W^*_{n,2,1}$ with probability $1 - O(n^{-1/2})$ at least, and it suffices to show that

$$\lim_{n \to \infty} \mathbf{E}(W^*_{n,2,1})_2 = \lambda^t, \quad t \ge 1. \tag{27}$$

This is obviously true for $t = 1$. Let $t \geq 2$. Combinatorially, $(W^*_{n,2,1})_t$ is the total number of ordered t-tuples of (vertex-disjoint) 2-witnesses. Given $r + s = t$, let us compute E_{rs}, the expected number of t-tuples containing r "2 rows, 1 column" (first kind) witnesses, and s "2 columns, 1 row" (second kind) witness. The r vertex-disjoint first kind of witnesses can be chosen in $\binom{n}{2r}\binom{n}{r}(2r-1)!!r!$ ways. (Indeed, once $2r$ rows and r columns are selected, we pair the rows in $(2r-1)!!$ ways and assign the formed r pairs to r columns in $r!$ ways.) Given any such choice, the s 2-nd witnesses, disjoint among themselves *and* from the r first kind witnesses, can be chosen in $\binom{n-r}{2s}\binom{n-2r}{s}(2s-1)!!s!$ ways. There are $t! = (r+s)!$ ways to order all $r + s$ witnesses. Hence $N_1(r, s)$, the total number of the ordered t-tuples of the "alleged" witnesses, is given by

$$\begin{aligned} N_1(r,s) &= \binom{n}{2r}\binom{n}{r}(2r-1)!!r!\binom{n-r}{2s}\binom{n-2r}{s}(2s-1)!!s!(r+s)! \\ &\sim \binom{t}{r}\frac{n^{3t}}{2^t}. \end{aligned}$$

Deleting $2r$ rows and $2s$ columns involved in first kind and second kind witnesses respectively produces a bipartite graph with $m-2t$ edges that meets the following conditions. (a) Every row (column) vertex not involved in the s 2-nd (in the r first) kind witnesses has degree at least 1. (b) No edge can be added to one of (just deleted) $r + s$ 2-witnesses to form a pair (K, L), such that $|K| = 3, |L| = 1$, $K \subset R, L \subset C$, or $K \subset C, L \subset R$, and $N(K) = L$. (This condition is necessary and sufficient for the $(r + s)$ 2-witnesses to be disjoint from all other 2-witnesses.) Denote the total number of such graphs by $N_2(r, s)$. Clearly $N_2(r, s) \leq \mathcal{N}_2(r, s)$, where $\mathcal{N}_2(r, s)$ is the total number of bipartite graphs with the condition (b) dropped. Using (7) with $r_1 = r_2 = \rho$, we have

$$\begin{aligned} \mathcal{N}_2(r,s) &\sim (m-2t)! \cdot \frac{(e^\rho-1)^{2n-3t}e^{t\rho}}{\rho^{2(m-2t)}} \\ &\times \left(e^{-\frac{\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}{2}} \cdot \mathbf{Pr}(R = m-2t)\mathbf{Pr}(S = m-2t) + O(e^{-\log^5 m})\right). \end{aligned}$$

Here $R = \sum_{i=1}^{n-2r} Y_i$, $S = \sum_{j=1}^{n-2s} Z_j$, $Y_i, Z_j = \text{Po}(\rho; \geq 1)$ for $1 \leq i \leq n-2r-s$, $1 \leq j \leq n-r-2s$, and $Y_i, Z_j = \text{Po}(\rho)$ for $n-2r-s < i \leq n-2r$, $n-r-2s < j \leq n-2s$. Using (10) for both local probabilities, we obtain that the second line in the above formula is asymptotic to

$$\exp\left(-\frac{(n\mathbf{E}(Y_1)_2)^2}{2m^2}\right) \cdot \frac{1}{2\pi n \mathbf{Var}(Y_1)}.$$

Thus

$$\frac{\mathcal{N}_2(r,s)}{N_1(n,m)} \sim m^{-2t}\rho^{4t}(e^\rho-1)^{-3t}e^{t\rho} \sim \left(\frac{e^{-c_n}}{4n^3}\right)^t. \tag{28}$$

Now

$$\mathcal{N}_2(r,s) - N_2(r,s) \leq r(n-2r-s)N_2^{(1)}(r,s) + s(n-r-2s)N_2^{(2)}(r,s);$$

here $N_2^{(1)}(r, s)$ ($N_2^{(2)}(r, s)$ resp.) is the total number of the remaining graphs, such that a particular row (column resp.) vertex is incident to a single column (row resp.) vertex, which happens to be one of the vertices from r first kind (s second kind resp.) witnesses. Consider $N_2^{(1)}(r, s)$. Deleting that row we get a graph with one less number of row vertices and one less number of edges. So, using (7) with $r_1 = r_2 = \rho$ and $e^\rho - 1 \sim e^\rho$, we obtain that

$$\begin{aligned} \frac{N_2^{(1)}(r,s)}{N_1(n,m)} &\leq_b \frac{\mathcal{N}_2(r,s)}{N_1(n,m)}\frac{\rho^2}{me^\rho}, \\ \frac{N_2^{(2)}(r,s)}{N_1(n,m)} &\leq_b \frac{\mathcal{N}_2(r,s)}{N_1(n,m)}\frac{\rho^2}{me^\rho}. \end{aligned}$$

Therefore

$$\frac{\mathcal{N}_2(r,s) - N_2(r,s)}{N_1(n,m)} \le_b \frac{n\rho^2}{me^{\rho}} \le \rho e^{-\rho} \to 0.$$

Collecting the pieces, we obtain that

$$E_{rs} \sim \binom{t}{r} \frac{n^{3t}}{2^t} \cdot \left(\frac{\rho^4}{m^2 e^{2\rho}}\right)^t \to \binom{t}{r} \left(\frac{e^{-c}}{8}\right)^t,$$

i.e.

$$E(W^*_{n,2,1})_t \to \sum_{r=0}^{t} \binom{t}{r} \left(\frac{e^{-c}}{8}\right)^t = \lambda^t.$$

Thus $W^*_{n,2,1}$ is in the limit Poisson (λ), and then so is $W_{n,2,1}$. Consequently, a perfect matching exists with the limiting probability equal

$$\lim_{n\to\infty} \mathbf{Pr}(W_{n,2,1} = 0) = e^{-\lambda} = \exp\left(-\frac{e^{-c}}{4}\right).$$

Case 3: $c_n \to -\infty$**,** $m > n$.
3a: $m \ge (\frac{1}{3} - \epsilon) n \log n$.
In this case, after with trivial modifications in the above derivation,

$$\mathbf{E}(W^*_{n,2,1})_2 \sim \left(\frac{e^{-c_n}}{4}\right)^2 \to \infty,$$

and, by Chebyshev's inequality,

$$\lim_{n\to\infty} \mathbf{Pr}(W^*_{n,2,1} > 0) = 1.$$

So, **whp**, a perfect matching does not exist.

3b: $m \le (1/3 - \epsilon) n \log n$, $m - n \to \infty$.
Note that $n\rho \to \infty$. Let X_n denote the total number of isolated trees with 2 row vertices and 1 column vertex. ($X_n > 0$ implies that there is no perfect matching.) If the X_n trees are deleted, the remaining graph has $n - 2t$ row vertices, $n - t$ column vertices, and $m - 2t$ edges, and every vertex has degree 1, at least. Evaluating the number of such graphs by (7), we easily obtain

$$\begin{aligned} \mathbf{E}(X_n)_t &\sim \binom{n}{2t}\binom{n}{t}(2t-1)!!(t!)^2 \frac{(m-2t)!}{m!} \frac{\rho^{4t}}{(e^{\rho}-1)^{3t}} \\ &\sim \left(\frac{m\rho e^{-3\rho}}{2}\right)^t, \end{aligned}$$

using the definition of ρ for the second equality. Also from this definition, $\rho \sim 2(m-n)/n$ if $\rho \to 0$, and $\rho \le m/n$ always. So, if $\rho \to 0$,

$$m\rho e^{-3\rho} \sim 2m(m-n)/n \ge 2(m-n) \to \infty,$$

and, if $\lim \rho > 0$, then

$$m\rho e^{-3\rho} \ge \rho n e^{-3m/n} \ge \rho n^{3\epsilon} \to \infty.$$

Thus

$$\mathbf{E}(X_n) \to \infty, \quad \mathbf{E}(X_n)_2 \sim \mathbf{E}^2(X_n),$$

so that (Chebyshev's inequality) $\mathbf{Pr}(X_n > 0) \to 1$. That is, **whp** there is no perfect matching.

3c: $\sigma := m - n > 0$ is fixed.
If we form $4n - 3m$ isolated edges, the remaining $3(m-n)$ row vertices and $3(m-n)$ column vertices can be partitioned into $2(m-n)$ trees of size 3, half of the trees each containing 2 row vertices and 1 column vertex, and another half - 1 row vertex and 2 column vertices. The total number of such bipartite graphs is

$$\begin{aligned} N^*(n,m) &= \binom{n}{4n-3m}^2 (4n-3m)! \cdot \left[\binom{3(m-n)}{2(m-n)}(2(m-n)-1)!!(m-n)!\right]^2 \\ &\sim \frac{(n!)^2}{(n-3\sigma)!2^{2\sigma}(\sigma!)^2}. \end{aligned} \tag{29}$$

As for $N_1(n,m)$, the total number of all bipartite graphs, by Corollary 3 and (109), it is given by

$$N_1(n,m) \sim m! \left(\frac{f_1(\rho)^n e^{-(m-n)}(m-n)^{m-n}/(m-n)!}{\rho^m}\right)^2 \exp\left(-\frac{(n\mathbf{E}(Y)_2)^2}{2m^2}\right),$$

where, using the definition of ρ,

$$\rho = \frac{2\sigma}{n}\left(1 - \frac{2\sigma}{3n} + O(\sigma^2/n^2)\right).$$

So, after simple computations,

$$N_1(n,m) \sim \frac{m!n^{2\sigma}}{2^{2\sigma}(\sigma!)^2}. \tag{30}$$

Since, for fixed σ,

$$\frac{(n!)^2}{(n-3\sigma)!} \sim m!n^{2\sigma},$$

it follows from (29) and (30) that, with probability approaching 1, the random graph has $2\sigma > 0$ isolated trees of size 3, thus no perfect matching exists.

Theorem 1 is proved completely. □

4 Proof of Theorem 2

We notice upfront that, for $\liminf m/n > 1$, **whp** the random graph $G_{n,m}^{\delta\geq 2}$ has an almost perfect matching, in a sense that

$$\lim_{n\to\infty} \mathbf{Pr}(\mu^*(G_{n,m}^{\delta\geq 2}) < \lfloor n/2 \rfloor - n^{0.2+\beta}) = 0, \tag{31}$$

for every $\beta > 0$. This follows from the analysis of Karp-Sipser matching algorithm (its Phase 2, to be precise) [16], given in [2]. The analysis of [2] shows that at most $n^{0.2+\beta}$ vertices that are left at the end of Phase 1, are not covered by the matching constructed in Phase 2. The random graph at the end of Phase 1 is uniform, subject to the number of vertices ν, and edges μ and $\delta \geq 2$. The analysis is robust with respect to these parameters and implies (31). So, loosely speaking, our task is to get rid of the term $-n^{0.2+\beta}$.

First we prove (Lemma 2) that, analogously to the bipartite case, a graph with minimum vertex degree 2 at least, which has no perfect matching, must contain a certain (witness) subgraph. This result is based on the ideas of Edmonds' matching algorithm, [19] (Section 7, Exer. 34). Conditioned by the proof of Theorem 1, one would expect to be able to show that **whp** the random graph in question does not contain such a witness. Indeed, our next Lemma 3 rules out (**whp**) all the witnesses of size ϵn at most, $\epsilon > 0$ being sufficiently small. As in the proof of Theorem 2, our argument consists of showing that the expected count of "small" witnesses is exponentially small. However, we have not being able to extend the proof to larger witnesses. Apparently, for sparse graphs in question, the expected count of witnesses can be exponentially large, even though the count itself is zero **whp**.

Not everything is lost however! As the next step we show (Lemma 4) that **whp** either the random graph has a perfect matching, or there are an^2 pairs of disjoint vertices such that adding anyone of these pairs to the edge set of the subgraph, obtained by deletion of isolated odd cycles, increases the maximum matching number. This fact and a coupling device, that allows to relate, approximately, the random graphs $G_{n,m_1}^{\delta\geq 2}$ and $G_{n,m}^{\delta\geq 2}$ to each other, enable us to prove certain monotonicity of the distribution of $\mu^*(G_{n,m}^{\delta\geq 2})$ as the function of m, Lemma 9. We combine this monotonicity property of the maximum matching number and (31) to complete the proof of Theorem 2.

4.1 Step 1. Profiling and counting the witnesses.

We begin with

Lemma 2. *Let $G = (V, E)$ be a graph with $\delta(G) \geq 2$, and with no isolated odd cycles, which does not have a perfect matching, i.e. $\mu^*(G) < \lfloor |V|/2 \rfloor$. For every $x \in V$ which is not covered by at least one maximum matching, there exists a witness $(K, L) = (K(x), L(x))$, $K, L \subset V$, such that*

(i) $|K| = |L| + 1$;

(ii) $N_G(K) = L$, $\qquad (N_G(K) = \{w \notin K : \exists (v, w) \in E_G, v \in K\})$;

(iii) $|E_G(K \cup L)| \geq |K| + |L| + 1$;

(iv) *each $v \in L$ has at least 2 neighbours in K;*

(v) *for every $y \in K$, there exists a maximum matching that does not cover y;*

(vi) *adding any (x, y), $y \in K(x)$, to E increases the size of a maximum matching.*

Proof Let $x \in V$ and let M be a maximum matching which does not cover x. Since $\mu^*(G) < \lfloor |V|/2 \rfloor$, there exists $s \neq x$ which is also left uncovered by M. Now let T be a tree of maximal size which is rooted at s and such that for each $v \in T$, the path from s to v in T is alternating with respect to M. Let K, L be the set of vertices at even and odd distance respectively from s in T. For every $y \in K$, we can switch edges on the even path from the root to y to obtain another maximum matching that does not cover y. Furthermore, no leaf of T is in L, since otherwise switching edges along the odd- length path to such a leaf we would have increased the size of the matching. Therefore all the vertices of T, except s, are covered by M. Next, if a neighbor u of a vertex from K is not in $K \cup L$, then u must be covered by M, which contradicts maximality of T. Therefore the pair (K, L) meets all the conditions, except possibly (iii). Using the fact that all the leaves of T are from K and that their neighbors must be in $K \cup L$, *and* $\delta(G) \geq 2$, we can assert that

$$|E_G(K \cup L)| \geq |E(T)| + 1 = |K| + |L|.$$

But if $|E_G(K \cup L)| = |K| + |L|$ then T consists of two even-length path, sharing the root s only, with the leaves forming an edge in G. Thus s has degree 2 in G, and $K \cup L$ induces an odd cycle in G. As there are no isolated odd cycles in G, there must be some edge (v,w), $v \in L$, $w \notin K \cup L$. Since v is covered by M, $(v,w) \notin M$ and, for some $x \notin K \cup L$, we have $(w,x) \in M$. It is easy to see how to alter M solely on $E(K \cup L)$ to obtain a maximum matching M' and the corresponding tree T' rooted at v instead of s. (Draw a picture!) The degree of v in T' is at least 3, so T' has at least 3 leaves and, for the corresponding $K = K(T')$, $L = L(T')$, the condition (iii) is satisfied, too. □

Turn to $G^{\delta\geq 2}_{n,m}$. Given $\epsilon > 0$, let $\mathcal{A}_n(\epsilon)$ denote the event that there exist K, L satisfying (i)–(iv), and such that $|K| \leq \epsilon n$. The following lemma implies that **whp** witnesses must be large.

Lemma 3. *Let* $\liminf m/n > 1$. *There exists an* $\epsilon > 0$ *such that* $\mathbf{Pr}(\mathcal{A}_n(\epsilon)) = O(n^{-1})$.

Proof First of all, using $\liminf m/n > 1$, we have, [23]: $N(n,m)$ the total number of graphs with minimum degree at least 2 is asymptotic to

$$N_0(n,m) = \frac{(2m-1)!!}{\sqrt{2\pi n \mathrm{Var} Z}} \cdot \frac{f_2(\rho)^n}{\rho^{2m}} \cdot \exp(-\hat{\rho}/2 - \hat{\rho}^2/4). \tag{32}$$

Here $\rho, \hat{\rho}$ satisfy

$$\frac{\rho f_1(\rho)}{f_2(\rho)} = \frac{2m}{n}, \quad \hat{\rho} = \frac{\rho f_0(\rho)}{f_1(\rho)}, \tag{33}$$

i.e. ρ is bounded away from 0 and ∞, and Z is Poisson(ρ), conditioned on $Z \geq 2$. In fact, for all $a, b, x > 0$,

$$N(a,b) \leq c^* \frac{(2b-1)!!}{\sqrt{nx}} \cdot \frac{f_2(x)^a}{x^{2b}}, \tag{34}$$

where c^* does not depend on a, b, x. (The attentive reader certainly notices direct analogy between these formulas and their counterparts for the bipartite case in Section 2.) The independent copies $Z_1, \dots, Z_n$ of Z provide an approximation to $\mathbf{deg}(\Gamma)$, the degree sequence of the random graph Γ, in the following sense:

$$\mathbf{Pr}(\mathbf{deg}(\Gamma) \in B) = O\big(n^{1/2}\mathbf{Pr}(\mathbf{Z} \in B)\big), \tag{35}$$

uniformly for all sets B of n-tuples. Consequently, if B is such that $\mathbf{Pr}(\mathbf{Z} \in B)$ is $O(n^{-b})$ for some $b > 1/2$, then $\mathbf{Pr}(\mathbf{deg}(\Gamma) \in B) = O(n^{-(b-1/2)})$, which goes to zero, too! A particular event B, which will come in handy, is defined as follows. Let $d(j) = d(j, \Gamma)$ denotes the j-th largest degree of Γ. Pick $a > e^{5+\rho}(h(\rho)+1)^2$ where $h(\rho) = \frac{2}{f_2(\rho)}$ and define $\ell(n,j) = \lceil \log \frac{e^a n}{j} \rceil$. Let us show that

$$\mathbf{Pr}(\exists j \in [1,n] : \, d(j) > \ell(n,j)) = O(n^{-1}). \tag{36}$$

To prove this, consider first $Z(j)$, the j-th largest among $Z_1, \dots, Z_n$. Clearly

$$\mathbf{Pr}(Z(j) > \ell(n,j)) \leq \binom{n}{j} \mathbf{Pr}^j(Z_1 > \ell(n,j)) \leq \exp\left(j \log \frac{en}{j} + j \log \mathbf{Pr}(Z_1 \geq \ell(n,j))\right),$$

and, using the definition of $\ell(n,j)$ and a,

$$\begin{aligned} \mathbf{Pr}(Z_1 > \ell(n,j)) &\leq h(\rho) \frac{\rho^{\ell(n,j)}}{\ell(n,j)!} \leq \exp\left(\log h(\rho) - \ell(n,j) \log \frac{\ell(n,j)}{e\rho}\right) \\ &\leq \exp\left(-a \log \frac{e^a n}{j}\right). \end{aligned}$$

Consequently

$$\mathbf{Pr}(Z(j) > \ell(n,j)) \leq \exp\left(-j(a-1)\log \frac{en}{j}\right),$$

so that

$$\mathbf{Pr}(\exists j \in [1,n] : Z(j) > \ell(n,j)) \le \sum_{j=1}^{n} \mathbf{Pr}(Z(j) > \ell(n,j)) = O(n^{-2}),$$

whence the probability in (36) is $O(n^{-3/2})$. Now, for a given vertex subset S,

$$\sum_{j \in S} d_j \le \sum_{j=1}^{|S|} d(j),$$

and on the event in (36)

$$\sum_{j=1}^{s} d(j) \le \sum_{j=1}^{s} \left\lceil \log \frac{e^a n}{j} \right\rceil \le (2+a)s + s \log \frac{n}{s}.$$

We conclude that

$$\mathbf{Pr}\big(\exists S \subset [n] \, : \, \sum_{j \in S} d_j > (2+a)|S| + |S| \log(n/|S|)\big) = O(n^{-1}). \tag{37}$$

This bound will be needed shortly.

Now, given $k \in [2, \epsilon n]$, let T_{μ,ν,ν_1} denote the total number of pairs (K, L) consisting of disjoint subsets $K, L \subset [n]$ such that $|K| = k$, $|L| = k - 1$, **(i)**–**(iv)** hold and μ, ν, ν_1 are given by

$$\begin{aligned} &|E(K)| + |E(L)| = \mu, \\ &|\{(u,w) \in E(\Gamma) : u \in K, w \in L\}| = \nu, \\ &|\{(u,w) \in E(\Gamma) : u \in L, w \in (K \cup L)^c\}| = \nu_1. \end{aligned}$$

Note that by **(iii)**

$$\nu + \mu \ge 2k. \tag{38}$$

We want to show that

$$\mathbf{Pr}\left(\sum_{2 \le k \le \epsilon n} \sum_{\mu,\nu,\nu_1} T_{\mu,\nu,\nu_1} > 0 \right) = O(n^{-1}),$$

provided that $\epsilon > 0$ is sufficiently small. By 37, we may and will confine ourselves to μ, ν, ν_1 such that

$$\mu, \nu, \nu_1 \le A(k + k \log(n/k)), \tag{39}$$

for a large enough constant A. All we need to show is that

$$\sum_{k \le \epsilon n} \sum_{\substack{\mu,\nu,\nu_1: \\ (39) \text{ holds}}} E_{\mu,\nu,\nu_1} = O(n^{-1}), \quad E_{\mu,\nu,\nu_1} := \mathbf{E}(T_{\mu,\nu,\nu_1}). \tag{40}$$

By symmetry,

$$E_{\mu,\nu,\nu_1} = \binom{n}{k, k-1, n-2k+1} P_{\mu,\nu,\nu_1}, \tag{41}$$

where P_{μ,ν,ν_1} is the probability that the subsets $K^* = \{1, \dots, k\}$ and $L^* = \{k+1, \dots, 2k-1\}$ form such a pair. To bound this probability we need to bound N_{μ,ν,ν_1}, the total number of graphs in question in which the pair $\{K^*, L^*\}$ has the prescribed properties.

Let $(\delta_j)_{j \in K^*}$, $(\delta_j)_{j \in L^*}$ and $(\delta_j)_{j \in (K^* \cup L^*)^c}$ be the degree sequences for subgraphs $G(K^*)$, $G(L^*)$ and $G((K^* \cup L^*)^c)$ respectively. For $j \in K^*$ ($j \in L^*$ resp.) let Δ_j denote the total number of neighbors of j in L^* (in K^* resp.). For $j \in L^*$ ($j \in (K^* \cup L^*)^c$ resp.) let ∂_j denote the total number of neighbors of j in $(K^* \cup L^*)^c$ (in L^* resp.). Then

$$\sum_{j \in K^* \cup L^*} \delta_j = 2\mu, \qquad \sum_{j \in (K^* \cup L^*)^c} \delta_j = 2(m - \mu - \nu - \nu_1),$$
$$\sum_{j \in K^*} \Delta_j = \sum_{j \in L^*} \Delta_j = \nu, \qquad \sum_{j \in L^*} \partial_j = \sum_{j \in (K^* \cup L^*)^c} \partial_j = \nu_1. \tag{42}$$

In addition,

$$\delta_j + \Delta_j \geq 2, \quad j \in K^* \cup L^*,$$
$$\delta_j + \partial_j \geq 2, \quad j \in (K^* \cup L^*)^c. \tag{43}$$

It is worth noticing that (43) is a relaxed version of the actual restrictions. Also, lumping together δ_j for $j \in K^*$ and $j \in L^*$, we effectively ignore the fact that the graphs $G(K^*)$ and $G(L^*)$ are disjoint.

Denoting the total number of graphs with the given $\mathbf{D} = (\boldsymbol{\delta}, \boldsymbol{\Delta}, \boldsymbol{\partial})$ by $N(\mathbf{D})$, and using the degree-dependent bounds for the counts of graphs, both general and bipartite, we obtain

$$\begin{aligned}
&N(\mathbf{D}) \leq \\
&\left((2\mu-1)!! \prod_{j \in K^* \cup L^*} \frac{1}{\delta_j!}\right) \times \left(\nu! \prod_{j \in K^* \cup L^*} \frac{1}{\Delta_j!}\right) \times \left(\nu_1! \prod_{j \in (K^* \cup L^*)^c} \frac{1}{\partial_j!} \prod_{j \in L^*} \frac{1}{\partial_j!}\right) \times \\
&\left((2(m-\mu-\nu-\nu_1)-1)!! \prod_{j \in (K^* \cup L^*)^c} \frac{1}{\delta_j!}\right) \\
&= (2\mu-1)!!\nu!\nu_1!(2(m-\mu-\nu-\nu_1)-1)!! \cdot \Phi(\mathbf{D}); \\
&\Phi(\mathbf{D}) = \prod_{j \in K^* \cup L^*} \frac{1}{\delta_j!\Delta_j!} \cdot \prod_{j \in L^*} \frac{1}{\partial_j!} \cdot \prod_{j \in (K^* \cup L^*)^c} \frac{1}{\partial_j!\delta_j!}.
\end{aligned} \tag{44}$$

Our task now is to evaluate S_{μ,ν,ν_1}, the sum of $\Phi(\mathbf{D})$, for all $\mathbf{D}$ that meet (42) and (43). To do so, let us first determine a *multivariate* generating function of $\Phi(\mathbf{D})$, for $\mathbf{D}$ satisfying (43) only:

$$\begin{aligned}
&\sum_{\mathbf{D} \text{ satisfies } (43)} y_1^{\sum_{j \in K^* \cup L^*} \delta_j} y_2^{\sum_{j \in K^* \cup L^*} \Delta_j} y_3^{\sum_{j \in L^*} \partial_j} y_4^{\sum_{j \in (K^* \cup L^*)^c} \partial_j} y_5^{\sum_{j \in (K^* \cup L^*)^c} \delta_j} \cdot \Phi(\mathbf{D}) \\
&= \left(\sum_{\delta+\Delta \geq 2} \frac{y_1^\delta y_2^\Delta}{\delta!\Delta!}\right)^{2k-1} \left(\sum_{\partial \geq 0} \frac{y_3^\partial}{\partial!}\right)^{k-1} \left(\sum_{\partial+\delta \geq 2} \frac{y_4^\partial y_5^\delta}{\partial!\delta!}\right)^{n-2k+1} \\
&= f_2(y_1+y_2)^{2k-1} f_0(y_3)^{k-1} f_2(y_4+y_5)^{n-2k+1};
\end{aligned} \tag{45}$$

So now S_{μ,ν,ν_1} is the coefficient of $y_1^{2\mu} y_2^{2\nu} (y_3 y_4)^{\nu_1} y_5^{2(m-\mu-\nu-\nu_1)}$ in the function on the right hand of (45). Using

$$[z_1^{a_1} z_2^{a_2}] F(z_1+z_2) = \binom{a_1+a_2}{a_1} [z^{a_1+a_2}] F(z),$$

and $f_0(y_3) = e^{y_3}$, we obtain then:

$$S_{\mu,\nu,\nu_1} = \binom{2(\mu+\nu)}{2\mu}\binom{2(m-\mu-\nu)-\nu_1}{\nu_1}\frac{(k-1)^{\nu_1}}{\nu_1!} \\ \times [x_1^{2(\mu+\nu)}]f_2(x_1)^{2k-1} \cdot [x_2^{2(m-\mu-\nu)-\nu_1}]f_2(x_2)^{n-2k+1}. \quad (46)$$

Here

$$[x_1^{2(\mu+\nu)}]f_2(x_1)^{2k-1} \le \frac{f_2(x_1)^{2k-1}}{x_1^{2(\mu+\nu)}}, \quad \forall\, x_1 > 0,$$

and we will see that a sufficiently small x_1 will do the job. We can write an analogous bound for the last factor in (46), and (in the light of (32) and the relative smallness of our parameters μ, ν, ν_1) $x_2 = \rho$ is a natural choice. In fact, we can do a bit better and get an extra factor $n^{-1/2}$, by applying the Cauchy (circular) contour formula, cf. (82), in combination with (83). Using

$$P_{\mu,\nu,\nu_1} = \frac{N_{\mu,\nu,\nu_1}}{N(n,m)}, \\ N_{\mu,\nu,\nu_1} = \sum_{\mathbf{D} \text{ satisfies } (42),(43)} N(\mathbf{D}),$$

(32),(44), and an inequality

$$\binom{2u-v}{v}(2(u-v)-1)!! \le \frac{(2u-1)!!}{v!},$$

we obtain then

$$P_{\mu,\nu,\nu_1} \le_b (2\mu-1)!!\nu!\binom{2(\mu+\nu)}{2\mu}\frac{(2(m-\mu-\nu)-1)!!}{(2m-1)!!} \\ \times \frac{f_2(x_1)^{2k-1}}{x_1^{2(\mu+\nu)}} \cdot \frac{\rho^{2(\mu+\nu)}}{f_2(\rho)^{2k-1}} \cdot \frac{(\rho(k-1))^{\nu_1}}{\nu_1!}. \quad (47)$$

Then since

$$\sum_{\nu_1 \ge 0} \frac{(\rho(k-1))^{\nu_1}}{\nu_1!} = e^{\rho(k-1)} < e^{\rho k},$$

we get the bound (call it $Q_{\mu,\nu}$) for $\sum_{\nu_1} P_{\mu,\nu,\nu_1}$, which is (47) with the last factor replaced by $e^{\rho k}$.

Next, for $\nu \ge \nu_0 = \nu_0(\mu) := \max\{2(k-1), 2k-\mu\}$, using (39),

$$\frac{Q_{\mu,\nu+1}}{Q_{\mu,\nu}} \le_b \frac{\rho^2}{x_1^2} \cdot \frac{\mu^2+\nu^2}{\nu m} \\ \le_b \frac{\rho^2}{x_1^2} \cdot \left(\frac{k^2 + k^2\log^2(n/k)}{km} + \frac{\nu}{m}\right) \\ \le_b \frac{\rho^2}{x_1^2} \cdot (\epsilon + \epsilon\log^2(1/\epsilon)) \\ \le 1/2,$$

if $k \le \epsilon n$ and $0 < \epsilon \le \epsilon_1(x_1)$ is chosen sufficiently small. For this choice of ϵ,

$$\sum_{\nu \ge \nu_0(\mu)} Q_{\mu,\nu} = O(Q_{\mu,\nu_0(\mu)}). \tag{48}$$

Furthermore, if $\mu \ge 2$ then $\nu_0 = \nu_0(\mu) = 2(k-1)$ and we have

$$\begin{aligned} \frac{Q_{\mu+1,\nu_0(\mu+1)}}{Q_{\mu,\nu_0(\mu)}} &\le_b \frac{\rho^2}{x_1^2} \cdot \frac{\mu^2 + k^2}{\mu m} \\ &\le_b \frac{\rho^2}{x_1^2} \cdot \left(\frac{k}{m} + \frac{k}{m} \log^2(n/k) \right) \frac{k}{\mu} \\ &\le_b \frac{\rho^2}{x_1^2} \cdot (\epsilon + \epsilon \log^2(1/\epsilon)) \cdot \frac{k}{\mu} \\ &\le 1/2, \end{aligned}$$

if $\mu \ge \epsilon^{1/2} k$, and $0 < \epsilon \le \epsilon_2(x_1) < \epsilon_1(x_1)$ is chosen sufficiently small. If so,

$$\sum_{\mu \ge 2} Q_{\mu,\nu_0} \le_b k \max_{\mu} \{ Q_{\mu,\nu_0} : 2 \le \mu \le k\epsilon^{1/2} \}. \tag{49}$$

To make the last bound explicit, we use $(2a-1)!! = (2a)!/(2^a a!)$ and the Stirling formula for factorials to bound, for $\nu = \nu_0$ and $2 \le \mu \le \epsilon^{1/2} k$, the combinatorial factors in (47) as follows:

$$\begin{aligned} (2\mu - 1)!! &\le_b k^2 \left(\frac{2\mu}{e} \right)^{\mu-2}; \\ \binom{2(\mu+\nu_0)}{2\mu} &\le_b \exp(O(k\epsilon^{1/2} \log \epsilon^{-1})); \\ \frac{(2(m-\mu-\nu_0)-1)!!}{(2m-1)!!} &\le_b (2m)^{-(\mu-2)-2k} \exp(O(\epsilon k)). \end{aligned}$$

Using these bounds and $k^2 \nu_0! \le (2k)!$, we obtain: for $2 \le \mu \le \epsilon^{1/2} k$ and $x_1 < \lambda$,

$$Q_{\mu,\nu_0} \le_b n^{1/2} (2k)! (2m)^{-2k} \left(\frac{f_2(x_1)}{x_1^2} \right)^{2k} \left(\frac{\rho^2}{f_2(\rho)} \right)^{2k} e^{\rho k} \left(\frac{\rho}{x_1} \right)^{O(\epsilon^{1/2} k)} \exp(O(k\epsilon^{1/2} \log \epsilon^{-1})). \tag{50}$$

If $\mu = 0$ then $\nu_0 = 2k$, and if $\mu = 1$ then $\nu_0 = 2k-1$. The direct computation shows that the bound (50) holds in these two remaining cases as well. So, collecting the pieces and using

$$\frac{(2k)!}{k!(k-1)!} \le_b k 2^{2k},$$

we get

$$\sum_{\mu,\nu,\nu_1} E_{\mu,\nu,\nu_1} \le_b n^{-1/2} q^{2k} \exp(O(k\epsilon^{1/2} \log \epsilon^{-1})), \tag{51}$$

where

$$q = 2 \frac{n}{2m} \cdot \frac{f_2(x_1)}{x_1^2} \cdot \frac{\rho^2}{f_2(\rho)} \cdot e^{\rho/2}. \tag{52}$$

Using (33), we transform (52) into

$$q = \frac{\rho}{e^{\rho/2} - e^{-\rho/2}} \cdot \frac{2 f_2(x_1)}{x_1^2}.$$

The first fraction is strictly less than, and bounded away from 1. (That's where the condition "$\liminf m/n > 1$" enters!). And the second fraction approaches 1, from above, when $x_1 \downarrow 0$. So we can pick x_1 small enough to make $\rho < 1$. For this choice of x_1, and the corresponding $\epsilon = \epsilon(x_1) < \epsilon_2(x_1)$, we have

$$q^2 \exp(O(\epsilon^{1/2} \log \epsilon^{-1})) \le q_1 := \frac{1+q^2}{2} < 1.$$

Then (51) implies that

$$\sum_{k \le \epsilon n} \sum_{\mu,\nu,\nu_1} E_{\mu\nu\nu_1} \le_b n^{-1/2} \sum_{k \ge 0} q_1^k = O(n^{-1/2}).$$

Thus (40) is completely proved, and so is Lemma 3. □

Note. Let X_n denote the total number of isolated odd cycles in $G_{n,m}^{\delta \ge 2}$, and let $\tilde{G}_n$ denote the random subgraph obtained by deletion of all X_n cycles in question. If $\mu(\tilde{G}_n) < \lfloor (n - X_n)/2 \rfloor$ then, by the previous lemma, $\tilde{G}_n$ must contain a witness (K, L). According to the last lemma, **whp** K has to be large, of size ϵn, at least.

4.2 Step 2. Using a witness to gainfully add new edges.

Introduce ∂E, the set of *non-edges* (x,y) of the random (sub)graph $\tilde{G}_n$, such that adding (x,y) to the edge set makes the maximum matching number increase by one.

Lemma 4.

$$\mathbf{Pr}(0 < |\partial E| < \epsilon^2 n^2/2) = O(n^{-1}).$$

Proof Suppose the event $\{|\partial E| > 0\} \cap \mathcal{A}_n(\epsilon)^c$ happens. Then, by Lemma 2, for every vertex $x \in \tilde{V} := V(\tilde{G}_n)$, not covered by at least one maximum matching (in $\tilde{G}_n$, needless to say), there exists a vertex set $K \subset \tilde{V}$ of cardinality ϵn or more, such that (1) $x \notin K$, $x \notin N_{\tilde{G}_n}(K)$; (2) adding any (x,y), $y \in K$, to the edge set of $\tilde{G}_n$ increases the maximum matching number; (3) for every vertex $y \in K$ there exists a maximum matching that does not cover y. This implies existence of the vertices $x_1, \dots, x_{\nu_n} \in \tilde{V}$, $(\nu_n := \lfloor \epsilon n \rfloor)$, such that for every x_j there is a corresponding vertex subset $Y_j \subset \tilde{V}$ satisfying the conditions: (1) $x_j \notin Y_j$, $x_j \notin N_{\tilde{G}_n}(Y_j)$; (2) for every $y \in Y_j$, adding (x_j, y) to the edge set increases the maximum matching number; (3) $|Y_j| \ge \nu_n$. Consequently the edge set of $\tilde{G}_n$ is missing at least

$$\sum_{j=1}^{\nu_n} (\nu_n - j + 1) = \frac{\nu_n(\nu_n + 1)}{2} \ge \frac{\epsilon^2 n^2}{2}$$

pairs (x,y) such that adding any such pair to the edge set would increase the maximum matching. Therefore

$$\{0 < |\partial E| \le \epsilon^2 n^2/2\} \subseteq \mathcal{A}_n(\epsilon),$$

and the claim follows from Lemma 3. □

Note. Paraphrasing Lemma 4, with probability $1 - O(n^{-1})$, the subgraph $\tilde{G}_n$ (equal $G_{n,m}^{\delta \ge 2}$ minus all its isolated odd cycles) either has a perfect matching or there are at least an^2 pairs of vertices $(u,v) \in \tilde{V} \times \tilde{V}$, $(u,v) \notin E(\tilde{G}_n)$, such that adding any such (u,v) would increase the matching number.

4.3 Step 3. Counting the isolated odd cycles.

Clearly, we need to determine the limiting distribution of X_n, the total number of isolated odd cycles in $G_{n,m}^{\delta\geq 2}$.

Lemma 5. *Let* $\liminf m/n > 1$. *Then* X_n *is, in the limit, Poisson* (λ)*, i.e.*

$$\mathbf{Pr}(X_n = j) - e^{-\lambda}\frac{\lambda^j}{j!}, \quad j \geq 0. \tag{53}$$

Here

$$\lambda := \frac{1}{4}\log\frac{1+\sigma}{1-\sigma} - \frac{\sigma}{2}, \quad \sigma = \frac{\rho}{e^{\rho}-1}, \tag{54}$$

and ρ *is defined in (33).*

Proof Let $X_{n,\ell}$ denote the total number of isolated odd cycles of length $\ell \geq 3$. Then, given $L > 3$,

$$\mathbf{E}\left(\sum_{\ell\geq L} X_{n,\ell}\right) = \sum_{\ell \geq L}\binom{n}{\ell}\cdot\frac{(\ell-1)!}{2}\frac{N(n-\ell, m-\ell)}{N(n,m)}. \tag{55}$$

Using (32) and (34) with $x = \rho$, and

$$\prod_{j=0}^{\ell-1}\frac{n-j}{2(m-j)-1} \leq \left(\frac{n}{2m-1}\right)^{\ell},$$

we see that the generic term in the sum is of order at most

$$\frac{1}{2\ell}\sqrt{\frac{n}{n-\ell}}\cdot\left(\frac{n}{2m}\cdot\frac{\rho^2}{f_2(\rho)}\right)^{\ell} = \frac{1}{2\ell}\cdot\sqrt{\frac{n}{n-\ell}}\cdot\sigma^{\ell}.$$

Since $\sigma < 1$, it easily follows then that $\mathbf{E}\left(\sum_{\ell\geq L} X_{n,\ell}\right) \to 0$ if $L = L(n) \to \infty$ however slowly. Consequently, **whp** there are no isolated odd cycles of length exceeding $L = L(n)$. Introduce

$$X_n^* = \sum_{\substack{\ell \leq L(n)\\ \ell \text{ odd}}} X_{n,\ell}.$$

Then, for every fixed $k \geq 1$,

$$\begin{aligned}
\mathbf{E}[(X_n^*)_k] &= \sum_{\ell\leq kL} R_{n,m}(\ell)\sum_{\substack{\ell_1,\dots,\ell_k\in[3,L]\\ \ell_1+\cdots+\ell_k=\ell\\ \ell_1,\dots,\ell_k \text{ odd}}}\prod_{j=1}^{k}\frac{(\ell_j-1)!}{2(\ell_j!)}\\
&= \sum_{\ell\leq kL} R_{n,m}(\ell)\cdot[x^{\ell}]\left(\sum_{j\in[3,L], j \text{ odd}}\frac{x^j}{2j}\right)^{K};\\
R_{n,m}(\ell) &:= \frac{n!N(n-\ell,m-\ell)}{(n-\ell)!N(n,m)}.
\end{aligned}$$

For $\ell \leq L(n)$ and $L(n) \to \infty$ sufficiently slowly, $N(n-\ell, m-\ell)$ is asymptotic the RHS in (32), with n and m replaced by $n-\ell$ and $m-\ell$. (The point here is that the difference between ρ and $\rho(\ell)$ corresponding to $n-\ell$, $m-\ell$ is of order $O(L/n)$, and this difference leads to an extra factor $\exp(O(L^2/n)) \to 1$, provided that $L = o(n^{1/2})$.) Consequently $R_{n,m}(\ell) \sim \sigma^\ell$, uniformly for $\ell \le L$. Therefore, using $\sigma < 1$,

$$\begin{aligned}
\mathbf{E}[(X_n^*)_k] \sim \quad & \textstyle\sum_{\ell \le kL} \sigma^\ell \cdot [x^\ell] \left(\displaystyle\sum_{j \in [3,L], j \text{ odd}} \frac{x^j}{2j} \right)^k \\
= \quad & \sum_{\ell \le kL} [x^\ell] \left(\sum_{j \in [3,L], j \text{ odd}} \frac{(\sigma x)^j}{2j} \right)^k \\
\sim \quad & \sum_{\ell} [x^\ell] \left(\sum_{j \in [3,L], j \text{ odd}} \frac{(\sigma x)^j}{2j} \right)^k \\
= \quad & \left(\sum_{j \in [3,L], j \text{ odd}} \frac{\sigma^j}{2j} \right)^k .
\end{aligned}$$

Thus X_n^* is in the limit Poisson with parameter

$$\sum_{j \in [3,L], j \text{ odd}} \frac{\sigma^j}{2j} = \frac{1}{4} \log \frac{1+\sigma}{1-\sigma} - \frac{\sigma}{2} + o(1) = \lambda + o(1).$$

Since $\mathbf{Pr}(X_n \ne X_n^*) \to 0$, the proof is complete. □

As a brief summary of Steps 1-3, we have established that **whp** $G_{n,m}^{\delta \ge 2}$ contains few isolated odd cycles, and upon deletion of these cycles we end up with a subgraph such that if it has no perfect matching, then there are of order n^2 non-edges, whose individual insertions would increase the maximum matching number.

To capitalize on these results, we need to find a way to compare the maximum matching number for two different values of m. And this is a serious challenge, since—unlike the Erdős-Rényi random graph $G(n,m)$ —we do not know of any construction which would have allowed to consider $G_{n,m_1}^{\delta \ge 2}$ as a subgraph of $G_{n,m_2}^{\delta \ge 2}$, for $m_1 < m_2$. The next step shows how such a coupling can be done asymptotically.

4.4 Step 4. An asymptotic coupling of $G_{n,m-\omega}^{\delta \ge 2}$ and $G_{n,m}^{\delta \ge 2}$.

Let $\omega = \lfloor r \log n \rfloor$ for some constant $r > 0$. Consider the bipartite graph Γ with vertex set bipartition $\mathcal{G}_{n,m-\omega}^{\delta \ge 2} + \mathcal{G}_{n,m}^{\delta \ge 2}$ and the edge set $E(\Gamma)$ defined by the condition: for $G \in \mathcal{G}_{n,m-\omega}^{\delta \ge 2}$ and $H \in \mathcal{G}_{n,m}^{\delta \ge 2}$, $(G,H) \in E(\Gamma)$ iff

$$E(G) \subseteq E(H) \text{ and } E(G) \setminus E(H) \text{ is a matching.}$$

(So $(G,H) \in E(\Gamma)$ iff $E(H)$ is obtained by adding to $E(G)$ ω independent edges.)

Consider the following experiment SAMPLE:

- Choose G randomly from $\mathcal{G}_{n,m-\omega}^{\delta \ge 2}$
- Add a random matching M, disjoint from $E(G)$, of size ω to obtain $H \in \mathcal{G}_{n,m}^{\delta \ge 2}$.

This induces a probability measure $\mathbf{Q}$ on $\mathcal{G}_{n,m}^{\delta \ge 2}$. Our task is to show that $\mathbf{Q}$ is *nearly* uniform.

For $v \in \mathcal{G}_{n-\omega}^{\delta \ge 2} + \mathcal{G}_{n,m}^{\delta \ge 2}$, let $d_\Gamma(v)$ denote the degree of v.

Lemma 6. *$G \in \mathcal{G}_{n,m-\omega}^{\delta \geq 2}$ implies*

$$\frac{(\binom{n}{2} - m - 2\omega n)^\omega}{\omega!} \leq d_\Gamma(G) \leq \binom{\binom{n}{2}}{\omega}.$$

Proof The RHS is obvious. For the LHS let us bound from below the number of *ordered* sequences $e_1, e_2, \ldots, e_\omega$ of ω edges which are not in $E(G)$, and form a matching. Observe that after choosing $e_1, e_2, \ldots, e_i$ we rule out at most $m - \omega + 2in$ choices for e_{i+1}. (The $m - \omega$ edges of G plus the further $\leq 2i(n-2)$ choices of new edges incident with $e_1, e_2, \ldots, e_i$). Thus there are always at least $\binom{n}{2} - m - 2\omega n$ choices for e_{i+1}. Dividing by $\omega!$ accounts for removing the ordering. □

Thus for n large and $G, G' \in \mathcal{G}_{n,m-\omega}^{\delta \geq 2}$,

$$\left| \frac{d_\Gamma(G)}{d_\Gamma(G')} - 1 \right| \leq \frac{4\omega^2}{n}. \tag{56}$$

We need to prove analogous estimates for the degrees $d_\Gamma(H)$, $H \in \mathcal{G}_{n,m}^{\delta \geq 2}$.

To this end, let $\Delta(H)$ denote the maximum vertex degree in H, and let $E_>(H)$ be the edges of H joining vertices of degree at least 3. (Why looking at $E_>(H)$? Well, if (G, H) is an edge in $\mathcal{G}$, and $e \in E(H) \setminus E(G)$, then other edges of H incident to e must already be in $E(G)$. So $E(H) \setminus E(G) \subseteq E_>(H)$.)

Lemma 7. *Let*

$$\theta = c^{-1} y^2, \text{ where } \frac{\rho(e^\rho - 1)}{e^\rho - 1 - y} = c, \quad c = \frac{2m}{n}.$$

If H is chosen uniformly at random from $\mathcal{G}_{n,m}^{\delta \geq 2}$ then **qs**[1]

(a)

$$\Delta(H) \leq \log n.$$

(b)

$$|E_>(H) - \theta n| = O(n^{1/2} \log n).$$

Proof Let $Z_1, Z_2, \ldots, Z_n$ be independent copies of $\text{Po}(\rho; \geq 2)$. Introduce the random set $S_{\mathbf{Z}}$ that contains Z_i (distinguishable) copies of the vertex i, $1 \leq i \leq n$; denote $s_{\mathbf{Z}} = |S_{\mathbf{Z}}|$. Given $S_{\mathbf{Z}}$, we choose uniformly at random "pairing" of all $s_{\mathbf{Z}}$ elements of $S_{\mathbf{Z}}$. A convenient way of generating such a pairing, is to choose a random permutation $\pi = (\pi_1, \ldots, \pi_{s_{\mathbf{Z}}})$ of $S_{\mathbf{Z}}$ and to form pairs $(\pi_1, \pi_2), (\pi_3, \pi_4), \ldots$. (When $s_{\mathbf{Z}}$ is odd, one vertex in $\sigma_{\mathbf{Z}}$ remains without a partner.) Conditional on $\{s_{\mathbf{Z}} = 2m\} \cap \{\text{pairing is graph- induced}\}$, the uniformly random pairing (permutation π) defines a uniformly random graph $H \in \mathcal{G}_{n,m}^{\delta \geq 2}$. Like the bipartite case in Section 2, we have that

$$\mathbf{Pr}(\mathcal{E}_0) = \mathbf{E}\left(F(\mathbf{Z}) \cdot 1_{\{s_{\mathbf{Z}} = 2m\}}\right), \quad s_{\mathbf{Z}} = \sum_{j=1}^n Z_j,$$

where

$$F(\mathbf{Z}) = \exp\bigl(-\eta(\mathbf{Z})/2 - \eta^2(\mathbf{Z})/4 + O(\max_j Z_j^4/m)\bigr), \quad \eta(\mathbf{Z}) = \frac{1}{2m} \sum_{j=1}^n Z_j(Z_j - 1),$$

[1]A sequence of events $\mathcal{E}_n$ is said to occur *quite surely* if $\mathbf{Pr}(\mathcal{E}_n) = 1 - O(n^{-K})$ for any $K > 0$.

cf. [23]. Implicit in (32), (33) is

$$\mathbf{Pr}(\mathcal{E}_0) = (1+o(1))\frac{\beta}{\sqrt{n}}, \quad \beta = \frac{\exp\left(-\hat\rho/2 - \hat\rho^2/4\right)}{\sqrt{2\pi \mathrm{Var}(Z)}}, \tag{57}$$

i.e. $\mathbf{Pr}(\mathcal{E}_0)$ is only polynomially small, of order $n^{-1/2}$ exactly. This implies that if $\{\mathbf{Z} \in A\}$, $A \subset \{\{2,3,\dots\}^n)$, is a **qs**event, then so is the event $\{\deg(H) \in A\}$, $\deg(H)$ denoting the degree sequence of $H \in \mathcal{G}^{\delta\geq 2}_{n,m}$. The part (a) follows then immediately since, for $L = \log n$,

$$\mathbf{Pr}(\max_j Z_j, \geq L) \leq n\mathbf{Pr}(Z_1 \geq L = \log n) = O(ny^L/L!),$$

which is $O(n^{-K})$ for any $K > 0$.

Turn to (b). Let W be the number of pairs (π_{2i-1}, π_{2i}) in the random permutation π of the multi-set $S_{\mathbf{Z}}$ such that both π_{2i-1} and π_{2i} are copies of the vertices of degree 3 or more. We know that, conditioned on the event $\mathcal{E}_0$, there is $W = E_>(H)$. And it is easy to see that

$$\mathbf{E}(W \mid \mathbf{Z}) = \frac{s_{\mathbf{Z},3}(s_{\mathbf{Z},3}-1)}{2(s_{\mathbf{Z}}-1)}, \tag{58}$$

where

$$s_{\mathbf{Z},3} = \sum_i Z_i \mathbf{I}_{\{Z_i>2\}}.$$

Now

$$\begin{aligned} \mathbf{E}(s_{\mathbf{Z}}) &= n\mathbf{E}Z_1 = n\frac{\rho(e^\rho-1)}{e^\rho-1-\rho} = 2m, && (59)\\ \mathbf{E}(s_{\mathbf{Z},3}) &= n(\mathbf{E}Z_1 - 2\mathbf{Pr}(Z_1=2)) = n\left(\frac{\rho(e^\rho-1)}{e^\rho-1-\rho} - \frac{\rho^2}{e^\rho-1-\rho}\right) && (60)\\ &= n\rho. && (61) \end{aligned}$$

And $s_{\mathbf{Z}}$, $s_{\mathbf{Z},3}$ are the sums of independent copies of Z and $\tilde{Z} := Z\mathbf{I}_{\{Z>2\}}$, respectively. Using the pgf's

$$\mathbf{E}(x^Z) = \frac{f_2(x\rho)}{f_2(\rho)}, \quad \mathbf{E}(x^{\tilde{Z}}) = \frac{\rho^2/2 + f_3(x\rho)}{f_2(\rho)},$$

(59), (61), in a standard (Chernoff-type) way, we obtain that **qs**

$$|s_{\mathbf{Z}} - 2m| \leq n^{1/2}\log n, \quad |s_{\mathbf{Z},3} - n\rho| \leq n^{1/2}\log n. \tag{62}$$

Denote the event in (62) by $\mathcal{E}_1$. Then $\mathcal{E}_1$ holds **qs**. It follows from (58) that, on the event $\mathcal{E}_1$,

$$\mathbf{E}(W \mid \mathbf{Z}) = \theta n + O(n^{1/2}\log n), \quad \theta := \frac{n}{2m}\rho^2. \tag{63}$$

Next we appeal to the Azuma-Hoeffding inequality to show that, conditional on $\mathbf{Z}$, W is tightly concentrated around $\mathbf{E}(W \mid \mathbf{Z})$. The A-H inequality applies since transposing any two elements of a permutation of $S_{\mathbf{Z}}$ may change W by at most 2, see Appendix B. So, for every $u > 0$,

$$\mathbf{Pr}(|W - \mathbf{E}(W \mid \mathbf{Z})| \geq u \mid \mathbf{Z}) \leq 2e^{-u^2/(8s_{\mathbf{Z}})}.$$

Removing the conditioning on $\mathbf{Z}$, and using the definition of the event $\mathcal{E}_1$, we obtain

$$\mathbf{Pr}(|W - \mathbf{E}(W \mid \mathbf{Z})| \geq u) \leq \mathbf{Pr}(\mathcal{E}_1^c) + 2e^{-u^2/(17m)}.$$

So, substituting $u = n^{1/2}\log n$ and using (63), we see that **qs**

$$|W - \theta n| \le An^{1/2}\log n,$$

if a constant A is sufficiently large. Recalling that $W = E_>(H)$ on the event $\mathcal{E}_0$, and that $\mathbf{Pr}(\mathcal{E}_0)$ is of order $n^{-1/2}$, we complete the proof of the part (b). □

Now let $\tilde{\mathcal{G}}$ be the set of $H \in \mathcal{G}_{n,m}^{\delta\ge 2}$ satisfying the conditions of the above lemma i.e.

- The number of edges joining two vertices of degree ≥ 3 is in the range $\theta n \pm An^{1/2}\log n$ for some constant $A > 0$.
- The maximum degree $\Delta(H) \le \log n$.

According to the lemma 7,

$$|\mathcal{G}_{n,m}^{\delta\ge 2} \setminus \tilde{\mathcal{G}}| \le |\tilde{\mathcal{G}}| n^{-K}, \quad \forall\, K > 0. \tag{64}$$

Note next that

Lemma 8. *$H \in \tilde{\mathcal{G}}$ implies*

$$\frac{(\theta n - An^{1/2}\log n - 2\omega\log n)^\omega}{\omega!} \le d_\Gamma(H) \le \binom{\theta n + An^{1/2}\log n}{\omega}.$$

Proof The upper bound follows from the earlier observation, namely that every edge among ω edges added to a graph $G \in \mathcal{G}_{n,m-\omega}^{\delta\ge 2}$ to obtain the graph H must connect two vertices which have degree 3 or more in H, and from the condition $H \in \tilde{\mathcal{G}}$. For the LHS, as in Lemma 6, let us bound from below the number of *ordered* sequences $e_1, e_2, \dots, e_\omega$ of ω edges which are contained in $E_>(H)$ and form a matching. Observe that after choosing $e_1, e_2, \dots, e_i$ we rule out at most $2i\Delta(H)$ choices for e_{i+1}, *since we have restricted ourselves to matchings.* Thus there are always at least $\theta n - An^{1/2}\log n - 2\omega\Delta$ choices for e_{i+1}. Dividing by $\omega!$ accounts for removing the the ordering. □

So for $H, H' \in \tilde{\mathcal{G}}$,

$$\left|\frac{d_\Gamma(H)}{d_\Gamma(H')} - 1\right| \le \frac{2A\omega\log n}{\theta n^{1/2}}. \tag{65}$$

Finally, for $H \in \mathcal{G}_{n,m}^{\delta\ge 2} \setminus \tilde{\mathcal{G}}$ and $H' \in \tilde{\mathcal{G}}$,

$$\frac{d_\Gamma(H)}{d_\Gamma(H')} \le \frac{\binom{m}{\omega}}{\frac{(\theta n - An^{1/2}\log n - 2\omega\log n)^\omega}{\omega!}} \le \left(\frac{c}{\theta}\right)^\omega, \quad c = \frac{2m}{n}, \tag{66}$$

as the total number of ways to delete a matching of size ω from $H \in \mathcal{G}_{n,m}^{\delta\ge 2}$ is $\binom{m}{\omega}$ at most.

Having proved the bounds (56), (65), (66), we can now show that the distribution $\mathbf{Q}$ on $\mathcal{G}_{n,m}^{\delta\ge 2}$, induced by the uniform distribution on $\mathcal{G}_{n,m-\omega}^{\delta\ge 2}$ *and* the SAMPLE, is nearly uniform itself.

Let $G_0 \in \mathcal{G}_{n,m-\omega}^{\delta\ge 2}$ be fixed. By (56), if $H \in \mathcal{G}_{n,m}^{\delta\ge 2}$ then

$$\begin{aligned}
\mathbf{Q}(H) &= \mathbf{Pr}(\text{SAMPLE chooses } H) \\
&= \frac{1}{|\mathcal{G}_{n,m-\omega}^{\delta\ge 2}|} \times \sum_{G:\,(G,H)\in E(\Gamma)} \frac{1}{d_\Gamma(G)} \\
&= \frac{1 + O(\omega^2/n)}{|\mathcal{B}_{n,m-\omega}^{\delta\ge 2}|} \cdot \frac{d_\Gamma(H)}{d_\Gamma(G_0)}.
\end{aligned} \tag{67}$$

From this relation, (65), and (66), it follows that

$$H, H' \in \tilde{\mathcal{G}} \quad \text{implies} \quad \left| \frac{\mathbf{Q}(H)}{\mathbf{Q}(H')} - 1 \right| \le \frac{3A\omega \log n}{\theta n^{1/2}}, \tag{68}$$

$$H \in \mathcal{G}_{n,m}^{\delta \ge 2} \setminus \tilde{\mathcal{G}},\, H' \in \tilde{\mathcal{G}} \quad \text{implies} \quad \frac{\mathbf{Q}(H)}{\mathbf{Q}(H')} \le \left(\frac{2c}{\theta} \right)^{\omega}. \tag{69}$$

Furthermore, invoking also

$$\sum_{G \in \mathcal{G}_{n,m-\omega}^{\delta \ge 2}} d_\Gamma(G) = \sum_{H \in \mathcal{G}_{n,m}^{\delta \ge 2}} d_\Gamma(H),$$

and picking $H' \in \tilde{\mathcal{G}}$, we obtain (see (56), (65)):

$$\frac{d_\Gamma(H')}{d_\Gamma(G_0)} \le \left(1 + \frac{4A\omega \log n}{\theta n^{1/2}} \right) \frac{|\mathcal{G}_{n,m-\omega}^{\delta \ge 2}|}{|\tilde{\mathcal{G}}|}. \tag{70}$$

Combining (64), (67), (69), and (70), we get: for every $K > 0$,

$$\begin{aligned} \mathbf{Q}(\mathcal{G}_{n,m}^{\delta \ge 2} \setminus \tilde{\mathcal{G}}) &\le \mathbf{Q}(H') \left(\frac{2c}{\theta} \right)^{\omega} \cdot n^{-2K} |\tilde{\mathcal{G}}| \\ &= \frac{1 + O(\omega^2/n)}{|\mathcal{G}_{n,m-\omega}^{\delta \ge 2}|} \cdot \frac{d_\Gamma(H')}{d_\Gamma(G_0)} \left(\frac{2c}{\theta} \right)^{\omega} n^{-2K} |\tilde{\mathcal{G}}| \\ &= O\big((2c/\theta)^{\omega} n^{-2K} \big) \\ &\le n^{-K}. \end{aligned} \tag{71}$$

(As $\omega = \lfloor r \log n \rfloor$, the last bound holds for $K > K(r)$.) Finally, since $\mathbf{Q}(\mathcal{G}_{n,m}^{\delta \ge 2}) = 1$, from (64), (71) and (68) we deduce that, for $H \in \tilde{\mathcal{G}}$,

$$\left| \mathbf{Q}(H) - \frac{1}{|\mathcal{G}_{n,m}^{\delta \ge 2}|} \right| \le \frac{1}{|\mathcal{G}_{n,m}^{\delta \ge 2}|} \cdot \frac{5A\omega \log n}{\theta n^{1/2}}. \tag{72}$$

This means that on the graph set $\tilde{\mathcal{G}}$ the probability measure $\mathbf{Q}$ is almost uniform.

4.5 Step 5. Asymptotic monotonicity of the tail distribution of the maximum matching number.

We will use $\tilde{G} = \tilde{G}(G)$ to denote the subgraph of G obtained by deletion of all isolated odd cycles in G, and $X(G)$ to denote the total number of these cycles. Clearly

$$\mu^*(\tilde{G}) \le \left\lfloor \frac{n - X(G)}{2} \right\rfloor, \quad \text{if } |V(G)| = n.$$

Let M stand for a generic value of the number of edges, and let $\mathbf{Pr}_M$ denote the uniform distribution on $\mathcal{G}_{n,M}^{\delta \ge 2}$. We will use $\mathbf{Q}_M$ to denote the probability distribution induced on $\mathcal{G}_{n,M}^{\delta \ge 2}$ by the uniform distribution $\mathbf{Pr}_{M-\omega}$ on $\mathcal{G}_{n,M-\omega}^{\delta \ge 2}$ via the SAMPLE procedure analyzed in Step 4. To shorten notations and to underscore dependence on M, we will write G_M and $\tilde{G}_M$ instead of $G_{n,M}^{\delta \ge 2}$ and $\tilde{G}(G_{n,M}^{\delta \ge 2})$ respectively, and X_M instead of $X(G_{n,M}^{\delta \ge 2})$. Introduce

$$\epsilon(M, \tau) = \mathbf{Pr}_M \left(\mu^*(G_M) < \left\lfloor \frac{n - X_M}{2} \right\rfloor - \tau \right).$$

Lemma 9. *Let $u \in (0,1)$ be so small that $(1-u)\liminf m/n > 1$. Then, for $M \in [(1-u)m, m]$ and $t \le t_n = n^{1/2}\log^{-3} n$,*

$$\epsilon(M, t-1) \le \epsilon(M-\omega, t) + \frac{6A\omega \log n}{n^{1/2}}. \tag{73}$$

Proof Let $H \in \mathcal{G}_{n,M}^{\delta\geq 2}$ be a right-side vertex in the graph Γ on the bipartition $\mathcal{G}_{n,M-\omega}^{\delta\geq 2} + \mathcal{G}_{n,M}^{\delta\geq 2}$. Suppose that

$$\mu^*(H) < \left\lfloor \frac{n - X(H)}{2} \right\rfloor - t + 1.$$

Then, for every $G \in N_\Gamma(H)$, either $\mu^*(G) < \lfloor (n - X(H))/2 \rfloor - t$, or $\mu^*(G) = \lfloor (n-X(H))/2 \rfloor - t$, in which case none of ω non-edges (u,v), added to $E(G)$, is in $\partial E(\tilde{G}(G))$. We know that

$$\mathbf{Pr}_{M-\omega}(1 \le |\partial E(\tilde{G}_{M-\omega})| \le an^2) = O(n^{-1}).$$

Besides, conditionally on $G_{M-\omega}$, the probability that none of ω random vertex disjoint non-edges belongs to $\partial E(\tilde{G}_{M-\omega})$, is bounded by

$$\left(1 - \frac{|\partial E(\tilde{G}_{M-\omega})| - \omega}{\binom{n}{2}}\right)^{\omega}.$$

And for $|\partial E(\tilde{G}_{M-\omega})| > an^2$, the last bound is at most $(1-a)^\omega \le n^{-1}$, if we pick r in $\omega = \lfloor r \log n \rfloor$ sufficiently large. Therefore

$$\begin{aligned} \mathbf{Q}_M\left(\mu^*(H) < \left\lfloor \frac{n - X(H)}{2} \right\rfloor - t + 1\right) &\le \mathbf{Pr}_{M-\omega}\left(\mu^*(G_{M-\omega}) < \left\lfloor \frac{n - X_{M-\omega}}{2} \right\rfloor - t\right) + O(n^{-1}) \\ &\le \epsilon(M-\omega, t) + \frac{b}{n}, \end{aligned}$$

for some absolute constant $b > 0$. So

$$\mathbf{Q}_M\left(\mu^*(H) < \left\lfloor \frac{n - X(H)}{2} \right\rfloor - t + 1 \text{ and } H \in \tilde{\mathcal{G}}\right) \le \epsilon(M-\omega, t) + \frac{b}{n},$$

and then, using (72),

$$\mathbf{Pr}_M\left(\mu^*(H) < \left\lfloor \frac{n - X(H)}{2} \right\rfloor - t + 1 \text{ and } H \in \tilde{\mathcal{G}}\right) \le \left(\epsilon(M-\omega, t) + \frac{b}{n}\right)\left(1 + \frac{5A\omega \log n}{\theta n^{1/2}}\right).$$

Therefore, recalling (71),

$$\begin{aligned} \mathbf{Pr}_M\left(\mu^*(H) < \left\lfloor \frac{n - X(H)}{2} \right\rfloor - t + 1\right) &\le \left(\epsilon(M-\omega, t) + \frac{b}{n}\right)\left(1 + \frac{5A\omega \log n}{\theta n^{1/2}}\right) + n^{-K} \\ &\le \epsilon(M-\omega, t) + \frac{6A\omega \log n}{\theta n^{1/2}}. \end{aligned}$$

In other words,

$$\epsilon(M, t-1) \le \epsilon(M-\omega, t) + \frac{6A\omega \log n}{\theta n^{1/2}},$$

for all $M \in [(1-u)m, m]$. □

4.6 Step 6. Completion of the proof of Theorem 2.

Iterating the inequality (73) $t \leq t_n$ times from $M = m$ and $t = 1$, we obtain

$$\epsilon(m, 0) \leq \epsilon(m - t\omega, t) + t \cdot \frac{7A\omega \log n}{n^{1/2}}. \tag{74}$$

Take $t = \lfloor n^{0.20+\beta} \rfloor$, $\beta > 0$ being small. Then, using the definition of $\epsilon(\cdot, \cdot)$ and (31), we obtain that $\epsilon(m - t\omega, t) \to 0$. And the second term on the RHS of (74) is $O(n^{-0.3+\beta} \log^2 n) = o(1)$, if $\beta < 0.3$.

We conclude that **whp** $\mu^*(G_{n,m})$ equals $\lfloor (n - X_m)/2 \rfloor$, so that $\tilde{G}(G_{n,m}^{\delta \geq 2})$ has a perfect matching. Since X_m is asymptotically Poisson(λ), this proves Theorem 2.

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A Enumerating bipartite graphs

Consider the bipartite graphs with vertex bipartition $R \cup C$, (Rows and Columns), $R = [\nu_1]$ and $C = [\nu_2]$. Given μ, the ν_1-tuple $\mathbf{a}$, and the ν_2-tuple $\mathbf{b}$ of nonnegative integers a_i, $i \in [\nu_1]$, b_j, $j \in [\nu_2]$, let $N(\mathbf{a}, \mathbf{b})$ denote the total number of the bipartite graphs with the row degree sequence $\mathbf{a}$ and the column degree sequence $\mathbf{b}$. Using the bipartite version of the pairing model, we see that

$$N(\mathbf{a}, \mathbf{b}) \le N^*(\mathbf{a}, \mathbf{b}); \quad N^*(\mathbf{a}, \mathbf{b}) := \frac{\mu!}{\prod_{i \in [\nu_1]} a_i! \cdot \prod_{j \in [\nu_2]} b_j!}. \tag{75}$$

The fudge factor, i.e. the ratio

$$F(\mathbf{a}, \mathbf{b}) = \frac{N(\mathbf{a}, \mathbf{b})}{N^*(\mathbf{a}, \mathbf{b})}, \tag{76}$$

is the probability that the uniformly random pairing is graph induced. A sharp asymptotic formula for $F(\mathbf{a}, \mathbf{b})$ has been a subject of many papers. A culmination point is [21] by McKay who proved that if $D^3/\mu \to 0$, D being the maximum degree, then

$$N(\mathbf{a},\mathbf{b}) = N^*(\mathbf{a},\mathbf{b}) \exp\left(-\frac{1}{2}\lambda(\mathbf{a})\lambda(\mathbf{b}) + O(D^3/\mu)\right), \tag{77}$$

$$\lambda(\mathbf{a}) := \frac{1}{\mu}\sum_{i\in[\nu_1]} a_i(a_i-1), \tag{78}$$

$$\lambda(\mathbf{b}) := \frac{1}{\mu}\sum_{j\in[\nu_2]} b_j(b_j-1), \tag{79}$$

The formulas (75) and (77) are instrumental in asymptotic evaluation (estimation) of the total number of bipartite graphs with a given number of edges and certain restrictions on the degree sequence.

Neglecting for now the fudge factor in (77),

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu},\mu) \le \sum_{\substack{a_i\ge c_i, b_j\ge d_j\\ \sum_i a_i=\sum_j b_j=\mu}} N^*(\mathbf{a},\mathbf{b}), \tag{80}$$

In order to rewrite (75) in a more manageable way, we observe that

$$\sum_{\mu_1,\mu_2\ge 0} x^{\mu_1}y^{\mu_2} \sum_{\substack{a_i\ge c_i, b_j\ge d_j\\ \sum_i a_i=\mu_1, \sum_j b_j=\mu_2}} \frac{1}{\prod\limits_{i\in[\nu_1]} a_i! \cdot \prod\limits_{j\in[\nu_2]} b_j!} = G_{\mathbf{c}}(x)H_{\mathbf{d}}(y);$$

Therefore (75) becomes

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu},\mu) \le \mu![x^\mu y^\mu]G_{\mathbf{c}}(x)H_{\mathbf{d}}(y) \tag{81}$$

$$= \mu!(2\pi i)^{-1}\oint\limits_{|x|=r_1} x^{-\mu-1}G_{\mathbf{c}}(x)\,dx\cdot(2\pi i)^{-1}\oint\limits_{|y|=r_2} y^{-\mu-1}H_{\mathbf{d}}(y)\,dy, \tag{82}$$

for all $r_1, r_2 > 0$. Using an inequality (Pittel [22])

$$|f_t(z)| \le f_t(|z|)\exp\left(-\frac{|z| - \text{Re } z}{t+1}\right), \tag{83}$$

(4), (5), and the fact that

$$|z| - \text{Re } z = r(1-\cos\theta) \ge cr\theta^2, \quad \text{when } z = re^{i\theta},\ \theta\in(-\pi,\pi],$$

we see from (82), after a straightforward estimation, that

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu},\mu) \le_b \mu!\frac{1}{\sqrt{r_1\sum_i(c_i+1)^{-1}}}\cdot\frac{G_{\mathbf{c}}(r_1)}{r_1^\mu}\times\frac{1}{\sqrt{r_2\sum_j(d_j+1)^{-1}}}\cdot\frac{H_{\mathbf{d}}(r_2)}{r_2^\mu}. \tag{84}$$

Here and elsewhere $A <_b B$ means that $A = O(B)$, uniformly for all feasible parameters that determine the values of A and B. In the sequel we consider only $\max_i c_i = O(1)$, $\max_j d_j = O(1)$, in which case the bound (84) simplifies to

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu},\mu) \le_b \mu!\cdot(\nu_1 r_1)^{-1/2}\frac{G_{\mathbf{c}}(r_1)}{r_1^\mu}\cdot(\nu_2 r_2)^{-1/2}\frac{H_{\mathbf{d}}(r_2)}{r_2^\mu}. \tag{85}$$

The task of determining the "best" values of r_1 and r_2 and incorporating the left-out fudge factor will be made easier by looking at the above through probabilistic lenses.

Fix $r_1, r_2 > 0$ and introduce the independent random variables Y_i, Z_j, with the distributions

$$\mathbf{Pr}(Y_i = \ell) = \frac{r_1^\ell/\ell!}{f_{c_i}(r_1)},\ (\ell \geq c_i), \tag{86}$$

$$\mathbf{Pr}(Z_j = \ell) = \frac{r_2^\ell/\ell!}{f_{d_j}(r_2)},\ (\ell \geq d_j), \tag{87}$$

so that, in distribution, Y_i is Poisson (r_1) conditioned on $\{\text{Poisson}(r_1) \geq c_i\}$, and Z_j is Poisson(r_2) conditioned on $\{\text{Poisson}(r_2) \geq d_j\}$. In short, $Y_i = \text{Po}(r_1; \geq c_i)$ and $Z_j = \text{Po}(r_2; \geq d_j)$. Now (81) can be rewritten as

$$\begin{aligned} N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) &\leq \mu! \times \frac{G_{\mathbf{c}}(r_1)}{r_1^\mu}[x^\mu]\frac{G_{\mathbf{c}}(xr_1)}{G_{\mathbf{c}}(r_1)} \cdot \frac{H_{\mathbf{d}}(r_1)}{r_1^\mu}[y^\mu]\frac{H_{\mathbf{d}}(yr_1)}{H_{\mathbf{d}}(r_1)} \\ &= \mu! \times \frac{G_{\mathbf{c}}(r_1)}{r_1^\mu}\mathbf{Pr}\left(\sum_i Y_i = \mu\right) \cdot \frac{H_{\mathbf{d}}(r_2)}{r_2^\mu}\mathbf{Pr}\left(\sum_j Z_j = \mu\right). \end{aligned} \tag{88}$$

Now the RHS expressions in (81), (82) and (88) are equal to each other and the RHS of inequality (85) bounds them all. Therefore,

$$\sup_\mu \mathbf{Pr}\left(\sum_i Y_i = \mu\right) \leq_b \frac{1}{\sqrt{\nu_1 r_1}}, \quad \sup_\mu \mathbf{Pr}\left(\sum_j Z_j = \mu\right) \leq_b \frac{1}{\sqrt{\nu_2 r_2}}. \tag{89}$$

Furthermore, (80) becomes equality when $N^*(\mathbf{a}, \mathbf{b})$ is replaced by $N(\mathbf{a}, \mathbf{b})$. So, analogously to (88),

$$\begin{aligned} N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) &= \frac{\mu!}{(r_1 r_2)^\mu} \sum_{\substack{a_i \geq c_i, b_j \geq d_j \\ \sum_i a_i = \mu_1, \sum_j b_j = \mu_2}} \prod_{i \in [\nu_1]} \frac{r_1^{a_i}}{a_i!} \prod_{j \in [\nu_2]} \frac{r_2^{b_j}}{b_j!} \\ &= \mu! \frac{G_{\mathbf{c}}(r_1) H_{\mathbf{d}}(r_2)}{(r_1 r_2)^\mu} \sum_{\substack{a_i \geq c_i, b_j \geq d_j \\ \sum_i a_i = \mu_1, \sum_j b_j = \mu_2}} \mathbf{Pr}(\mathbf{Y} = \mathbf{a}, \mathbf{Z} = \mathbf{b}) F(\mathbf{a}, \mathbf{b}) \\ &= \mu! \frac{G_{\mathbf{c}}(r_1) H_{\mathbf{d}}(r_2)}{(r_1 r_2)^\mu} \cdot \mathbf{E}\left(F(\mathbf{Y}, \mathbf{Z}) \cdot 1_{\{\sum_i Y_i = \mu\}} 1_{\{\sum_j Z_j = \mu\}}\right), \end{aligned} \tag{90}$$

where $F(\cdot, \cdot)$ is defined in (76). To make this formula useful, we need to show that, for a proper choice of r_1, r_2, asymptotically we can replace $\lambda(\mathbf{Y})\lambda(\mathbf{Z})$ in the formula (77) by $\mathbf{E}(\lambda(\mathbf{Y}))\mathbf{E}(\lambda(\mathbf{Z})))$.

From now on let us assume that and

$$\mu^{-1} \leq_b r_1, r_2 \leq_b \log \mu \text{ and that } \nu_1, \nu_2 = O(\mu). \tag{91}$$

Since $\max_i c_i$, $\max_j d_j$ are both $O(1)$, using the definition of Y_i, Z_j and the conditions on ν_1, ν_2 and r_1, r_2, we have: for $0 < \alpha' < \alpha$,

$$\begin{aligned} \mathbf{Pr}(\max\{\max_i Y_i, \max_j Z_j\} \geq \mu^\alpha) &\leq \sum_i \mathbf{Pr}(Y_i \geq \mu^\alpha) + \sum_j \mathbf{Pr}(Z_j \geq \mu^\alpha) \\ &\leq e^{-\mu^{\alpha'}}. \end{aligned}$$

Therefore, for $\alpha < 1/3$, $\mathbf{E}_{\boldsymbol{\nu},\mu}$, the expected value in (90), is given by

$$\begin{aligned} \mathbf{E}_{\boldsymbol{\nu},\mu} &= (1+O(\mu^{-1+3\alpha}))\mathbf{E}^*_{\boldsymbol{\nu},\mu} + O(e^{-\mu^{\alpha'}}); && (92)\\ \mathbf{E}^*_{\boldsymbol{\nu},\mu} &= \mathbf{E}\left(F^*(\mathbf{Y},\mathbf{Z})\cdot 1_{\{\sum_i Y_i=\mu\}}1_{\{\sum_j Z_j=\mu\}}\right); && (93)\\ F^*(\mathbf{a},\mathbf{b}) :&= \exp\left(-\frac{1}{2}\lambda(\mathbf{a})\lambda(\mathbf{b})\right), && (94) \end{aligned}$$

see (77). In particular, see (89),

$$\mathbf{E}^*_{\boldsymbol{\nu},\mu} \le_b (\nu_1\nu_2 r_1 r_2)^{-1/2}.$$

Let us estimate the effect of replacing $\lambda(\mathbf{Y}), \lambda(\mathbf{Z})$ in (94) by their expected values. To this end, let us introduce

$$U_i = (Y_i)_2 - \mathbf{E}((Y_i)_2), \quad V_j = (Z_j)_2 - \mathbf{E}((Z_j)_2).$$

Simple computation shows that $\mathbf{E}((Y_i)_2)$ is of order $O(1+r_i^2)$, whence of order $O(\log^2\mu)$, and likewise $\mathbf{E}(Z_j(Z_j-1)) = O(\log^2\mu)$. From $\lambda(\mathbf{Y}) = \mu^{-1}\sum_i (Y_i)_2$, $\lambda(\mathbf{Z}) = \mu^{-1}\sum_j (Z_j)_2$, it follows then that $\mathbf{E}(\lambda(\mathbf{Y}))$, $\mathbf{E}(\lambda(\mathbf{Z})) = O(\log^2\mu)$ and that after using the expansion

$$ab - \bar{a}\bar{b} = (a-\bar{a})(b-\bar{b}) + \bar{a}(b-\bar{b}) + \bar{b}(a-\bar{a})$$

we have

$$\begin{aligned} |\lambda(\mathbf{Y})\lambda(\mathbf{Z}) - \mathbf{E}(\lambda(\mathbf{Y}))\mathbf{E}(\lambda(\mathbf{Z}))| &\le_b (\log^2\mu)\Delta(\mathbf{Y},\mathbf{Z}) + \Delta^2(\mathbf{Y},\mathbf{Z}), && (95)\\ \Delta(\mathbf{Y},\mathbf{Z}) &= |\lambda(\mathbf{Y}) - \mathbf{E}(\lambda(\mathbf{Y}))| + |\lambda(\mathbf{Z}) - \mathbf{E}(\lambda(\mathbf{Z}))|. \end{aligned}$$

Therefore, if we replace $F^*(\mathbf{Y},\mathbf{Z})$ in (93) by $\exp(-\frac{1}{2}\mathbf{E}(\lambda(\mathbf{Y}))\mathbf{E}(\lambda(\mathbf{Z})))$, then the compensating factor is $\exp(O(\log^2\mu\Delta(\mathbf{Y},\mathbf{Z}) + \Delta^2(\mathbf{Y},\mathbf{Z})))$. Furthermore, setting $u = \log^{10}\mu$, we estimate

$$\begin{aligned} \mathbf{Pr}(|U_i| \ge u) &\le \sum_{\ell^2\ge u} \frac{r_1^\ell/\ell!}{f_{c_i}(r_1)}\\ &\le_b r_1^{-c_i}\left(\frac{er_1}{\log^5\mu}\right)^{\log^5\mu}\\ &\le \exp(-\Omega(\log^5\mu)). \end{aligned}$$

Likewise

$$\begin{aligned} \mathbf{E}(U_i; |U_i| \ge u) &= \sum_{\ell:(\ell)_2 - \mathbf{E}((Y_i)_2)\ge u} [(\ell)_2 - \mathbf{E}((Y_i)_2)]\frac{r_1^\ell/\ell!}{f_{c_i}(r_1)}\\ &\le_b r_1^{2-c_i}\left(\frac{2er_1}{\log^5\mu}\right)^{\frac{1}{2}\log^5\mu}\\ &\le \exp(-\Omega(\log^5\mu)). && (96) \end{aligned}$$

Let $\mathcal{U}_i = U_i\mathbf{I}_{\{|U_i|<u\}}$, so that $|\mathcal{U}_i| \le u$. Then (Azuma-Hoeffding inequality), for every $t > 0$,

$$\mathbf{Pr}\left(\left|\sum_i (\mathcal{U}_i - \mathbf{E}\mathcal{U}_i)\right| \ge t\right) \le 2\exp\left(-\frac{t^2}{2u^2\nu_1}\right).$$

Since $\mathbf{E}U_i = 0$, from (96) we have

$$\left|\sum_i \mathbf{E}\mathcal{U}_i\right| = \left|\sum_i \mathbf{E}(U_i; |U_i| \ge u)\right| \le \exp(-\Omega(\log^5\mu)).$$

Therefore, for $t \geq 1$,

$$\begin{aligned}
\mathbf{Pr}\left(\left|\sum_i U_i\right| \geq t\right) &\leq \sum_i \mathbf{Pr}(|U_i| \geq u) + \mathbf{Pr}\left(\left|\sum_i \mathcal{U}_i\right| \geq t\right) \\
&\leq \sum_i \mathbf{Pr}(|U_i| \geq u) + \mathbf{Pr}\left(\left|\sum_i (\mathcal{U}_i - \mathbf{E}\mathcal{U}_i)\right| \geq t - \left|\sum_i \mathbf{E}\mathcal{U}_i\right|\right) \\
&\leq_b \exp(-\Omega(\log^5 \mu)) + \exp\left(-\frac{\left(t - \exp(-\Omega(\log^5 \mu))\right)^2}{2u^2\nu_1}\right) \\
&= \exp(-\Omega(\log^5 \mu)), \qquad (97)
\end{aligned}$$

the latter inequality holding if $t = \mu^{1/2}\log^5 \mu$. An analogous inequality holds for $\sum_j V_j$. Equivalently

$$\begin{aligned}
\mathbf{Pr}\left(|\lambda(\mathbf{Y}) - \mathbf{E}(\lambda(\mathbf{Y}))| \geq \mu^{-1/2}\log^{10}\mu\right) &\leq \exp(-\Omega(\log^5 \mu)), \\
\mathbf{Pr}\left(|\lambda(\mathbf{Z}) - \mathbf{E}(\lambda(\mathbf{Z}))| \geq \mu^{-1/2}\log^{10}\mu\right) &\leq \exp(-\Omega(\log^5 \mu)).
\end{aligned}$$

Combining these bounds with (95) and (93), and denoting $R = \sum_i Y_i$, $S = \sum_j Z_j$, we get

$$\begin{aligned}
\mathbf{E}^*_{\boldsymbol{\nu},\mu} &= (1 + O(\mu^{-1/2}\log^{12}\mu))e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}\mathbf{Pr}\,(R = \mu)\,\mathbf{Pr}\,(S = \mu) + D_{\boldsymbol{\nu},\mu}; \qquad (98) \\
|D_{\boldsymbol{\nu},\mu}| &\leq_b \exp(-\Omega(\log^5 \mu)). \qquad (99)
\end{aligned}$$

In (98), the exponential factor is $\exp(-O(\log^4 \mu))$, and, by (89) and the conditions on μ, ν_i, r_i, the product of the probabilities is of order $(\nu_1 r_1 \nu_2 r_2)^{1/2}$, the latter being $\Omega((\mu \log \mu)^{-1})$. The resulting bound makes the remainder $D_{\boldsymbol{\nu},\mu}$ relatively negligible, so that

$$\mathbf{E}^*_{\boldsymbol{\nu},\mu} \leq_b \frac{e^{-\frac{1}{2}\mathbf{E}\lambda(\mathbf{Y})\mathbf{E}\lambda(\mathbf{Z})}}{(\nu_1 r_1 \nu_2 r_2)^{1/2}}.$$

The power of this bound is due to wide range of the parameters r_i for which it holds. However, we will also need an asymptotic formula for $\mathbf{E}^*_{\boldsymbol{\nu},\mu}$, and this requires asymptotic formulas for the local probabilities, rather than their upper bounds.

Intuitively, we stand a better chance of achieving this goal when the parameters r_1, r_2 are such that the events $\{\sum_i Y_i = \mu\}$ and $\{\sum_j Z_j = \mu\}$ have "sizeable" probabilities. What better candidates than $r_1 = \rho_1$ and $r_2 = \rho_2$ for which

$$\sum_{i=1}^{\nu_1} \mathbf{E}Y_i = \mu, \quad \sum_{j=1}^{\nu_2} \mathbf{E}Z_j = \mu.$$

Explicitly, using (86) and (87), ρ_1 and ρ_2 are the roots of

$$\sum_{i=1}^{\nu_1} \frac{x f_{c_i-1}(x)}{f_{c_i}(x)} = \mu, \qquad (100)$$

and

$$\sum_{j=1}^{\nu_2} \frac{x f_{d_j-1}(x)}{f_{d_j}(x)} = \mu, \qquad (101)$$

respectively; ($f_t(z) := e^z$, for $t \leq 0$). For $x \to 0$, the LHS of (100) and (101) approach $\sum_i c_i \leq \mu$ and $\sum_j d_j \leq \mu$, respectively. Each LHS is strictly increasing, asymptotic to $\nu_1 x$ and $\nu_2 x$

respectively, as $x \to \infty$. Assuming that $\sum_i c_i < \mu$ and $\sum_j d_j < \mu$, we see that the positive roots ρ_1 and ρ_2 exist uniquely, and that $\rho_i < \mu/\nu_i$. Assuming from now that $\mu = O(\nu_i \log \nu_i)$, $i = 1, 2$, we obtain that $\rho_i = O(\log \mu)$ which puts ρ_1 and ρ_2 into the set of feasible (meeting (91)) r_1 and r_2, respectively.

How do the probabilities in (98) behave if $r_1 = \rho_1, r_2 = \rho_2$? For $Y_i = \text{Po}(\rho; \geq 2)$ and $\mu = \nu \mathbf{E}(Y)$, it was proved in [2] that

$$\mathbf{Pr}\left(\sum_{i=1}^{\nu} Y_i = \mu + a\right) = \frac{1 + O(a^2(\rho\nu)^{-1})}{(2\pi\nu \mathbf{Var}(Y))^{1/2}},$$

provided that $\rho\nu \to \infty$, and $a^2/(\rho\nu) \to 0$. (The condition $\rho\nu \to \infty$ is equivalent to $\nu\mathbf{Var}(Y) \to \infty$ since $\mathbf{Var}(Y) = \Theta(\rho)$). Suppose that in the present context $\nu_1\rho_1 \to \infty$, i.e. $\sum_i \mathbf{Var}(Y_i) \to \infty$, which is equivalent to $\mu - \sum_i c_i \to \infty$. Only simple modifications of the proof in [2] are needed to prove

Lemma 10. *If $\sum_i \mathbf{Var}(Y_i) \to \infty$ and the c_i are uniformly bounded, then*

$$\mathbf{Pr}\left(\sum_{i=1}^{\nu_1} Y_i = \mu + a\right) = \frac{1 + O(a^2(\nu_1\rho_1)^{-1})}{(2\pi \sum_i \mathbf{Var}(Y_i))^{1/2}},$$

if $a^2(\nu_1\rho_1)^{-1} \to 0$. An analogous formula holds for $\mathbf{Pr}\left(\sum_j Z_j = \mu + a\right)$.

Proof Let $W = \sum_\ell Y_\ell$. As usual, we start with the inversion formula

$$\begin{aligned} \mathbf{Pr}\,(W = \tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\tau x}\mathbf{E}\left(e^{ix\sum_\ell Y_\ell}\right) dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\tau x}\prod_{\ell=1}^{\nu_1}\mathbf{E}(e^{ixY_\ell})^v\, dx, \end{aligned} \tag{102}$$

where $\tau = \mu + a$. Let

$$\Sigma_1 = \sum_\ell \frac{\rho_1}{c_\ell + 1} = \Theta(\nu_1\rho_1)$$

and consider first $|x| \geq \Sigma_1^{-5/12}$. Using inequality (83) we estimate

$$\begin{aligned} \frac{1}{2\pi}\int_{|x|\geq\Sigma_1^{-5/12}}\left|e^{-i\tau x}\prod_{\ell=1}^{\nu_1}\left(\frac{f_{c_\ell}(e^{ix}\rho_1)}{f_{c_\ell}(\rho_1)}\right)\right| dx &\leq \frac{1}{2\pi}\int_{|x|\geq\Sigma_1^{-5/12}}\prod_{\ell=1}^{\nu_1} e^{\rho_1(\cos x - 1)/(c_\ell+1)}\, dx \\ &\leq e^{-\Sigma_1^{1/6}/3}. \end{aligned} \tag{103}$$

For $|x| \leq \Sigma_1^{-5/12}$, putting $\eta = \rho_1 e^{ix}$ and using $\sum_\ell \rho_1 f'_{c_\ell}(\rho_1)/f_{c_\ell}(\rho_1) = \mu$, $d/dx = i\eta d/d\eta$ we expand as a Taylor series around $x = 0$ to obtain

$$\begin{aligned} -i\tau x + \sum_\ell \log\left(\frac{f_{c_\ell}(e^{ix}\rho_1)}{fc_\ell(\rho_1)}\right) &= -iax - \frac{x^2}{2}\sum_\ell \mathcal{D}\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\bigg|_{\eta=\rho_1} \\ &\quad -\frac{ix^3}{3!}\sum_\ell \mathcal{D}^2\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\bigg|_{\eta=\rho_1} \\ &\quad +O\left[x^4\sum_\ell \mathcal{D}^3\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\bigg|_{\eta=\tilde{\eta}}\right]; \end{aligned} \tag{104}$$

here $\tilde{\eta} = \rho_1 e^{i\tilde{x}}$, with $\tilde{x}$ being between 0 and x, and $\mathcal{D} = \eta(d/d\eta)$. Now, the coefficients of $x^2/2$, $x^3/3!$ and x^4 are $\mathbf{Var}(W)$, $O\left(\mathbf{Var}(W)\right)$, $O\left(\mathbf{Var}(W)\right)$ respectively, and $\mathbf{Var}(W)$ is of order Σ_1. So the second and the third terms in (104) are $o(1)$ uniformly for $|x| \le \Sigma_1^{-5/12}$. Therefore

$$\frac{1}{2\pi}\int_{|x|\le\Sigma_1^{-5/12}} = \int_1 + \int_2 + \int_3, \tag{105}$$

where

$$\begin{aligned}\int_1 &= \frac{1}{2\pi}\int_{|x|\le\Sigma_1^{-5/12}} e^{-iax-\mathbf{Var}(W)x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi\mathbf{Var}(W)}} + O\left(\frac{a^2+1}{\Sigma_1^{3/2}}\right), \end{aligned}\tag{106}$$

$$\begin{aligned}\int_2 &= O\left(\sum_\ell \mathcal{D}^2\left(\frac{\rho_1 f'_{c_\ell}(\rho_1)}{f(\rho_1)}\right)\int_{|x|\le\Sigma_1^{-5/12}} x^3 e^{-\mathbf{Var}(W)x^2/2}\,dx\right) \\ &= O\left(\Sigma_1\int_{|x|\ge\Sigma_1^{-5/12}} |x|^3 e^{-\mathbf{Var}(W)x^2/2}\,dx\right) \\ &= O(e^{-\alpha\Sigma_1^{1/6}}), \end{aligned}\tag{107}$$

($\alpha > 0$ is an absolute constant), and

$$\begin{aligned}\int_3 &= O\left[\Sigma_1\int_{|x|\le\Sigma_1^{-5/12}} x^4 e^{-\mathbf{Var}(W)x^2/2}\,dx\right] \\ &= O\left(\frac{1}{\Sigma_1^{3/2}}\right). \end{aligned}\tag{108}$$

Using (102)-(108), we arrive at

$$\mathbf{Pr}\left(W=\tau\right) = \frac{1}{\sqrt{2\pi v\mathbf{Var}(W)}} \times \left[1 + O\left(\frac{a^2+1}{\Sigma_1}\right)\right].$$

□

Suppose, say, that $\nu_1\rho_1$ is bounded, or equivalently that $\sigma_1 := \mu_1 - \sum_i c_i = O(1)$. Extending an argument in [23] (which covers the case of identically distributed Y_i), we can show that

$$\mathbf{Pr}\left(R=\mu\right) = (1+O(\nu_1^{-1}))e^{-\sigma_1}\frac{\sigma_1^{\sigma_1}}{\sigma_1!}. \tag{109}$$

An analogous relation holds for $\mathbf{Pr}(S=\mu)$ if $\sigma_2 := \mu - \sum_j d_j = O(1)$. Clearly then, regardless of the behavior of σ_1, σ_2, in (98) the remainder term $D_{\boldsymbol{\nu},\mu}$ is negligible compared to the explicit term. □

B Concentration of W.

We need to prove the following result.

Let S be a set with $|S| = N$. Let Ω be the set of $N!$ permutations of S. Let ω be chosen uniformly from Ω.

Let $Z = Z(\omega)$ be such that $|Z(\omega) - Z(\omega')| \le 1$ when ω' is obtained from ω by interchanging two elements of the permutation.

Lemma 11.

$$\mathbf{Pr}(|Z - \mathbf{E}Z| \geq t) \leq 2e^{-2t^2/N}.$$

Proof For a fixed sequence permutation $(x_1, x_2, \ldots, x_N)$ and $0 \leq i \leq N$ let

$$Z_i(x_1, x_2, \ldots, x_i) = \mathbf{E}(Z \mid \omega_j = x_j, 1 \leq j \leq i).$$

Clearly the sequence $Z_0, Z_1, \ldots, Z_N$ is a martingale. To apply the Azuma -Hoeffding inequality, we need to show that

$$|Z_i(x_1, x_2, \ldots, x_i) - Z_i(x_1, x_2, \ldots, x_{i-1}, x_i')| \leq 1 \tag{110}$$

for all i-tuples $(x_1, x_2, \ldots, x_i)$ with distinct components, and $x_i' \neq x_1, \ldots, x_{i-1}$. (Indeed, the inequality (110) readily implies that $|Z_{i+1} - Z_i| \leq 1$.)

Consider

$$\Omega_1 = \{\omega \in \Omega : \ \omega_j = x_j, 1 \leq j \leq i\}$$

and

$$\Omega_1' = \{\omega \in \Omega : \ \omega_j = x_j, 1 \leq j \leq i-1, \omega_i = x_i'\}$$

and the map $f : \Omega_1 \to \Omega_1'$ defined as follows. If $\omega = x_1 x_2 \ldots x_{i-1} x_i y_{i+1} \ldots y_N$ and $y_j = x_i'$ then $f(\omega) = x_1 x_2 \ldots x_{i-1} x_i' y_{i+1} \ldots y_{j-1} x_i y_{j+1} \ldots y_N$. Observe that f is a bijection and that

$$|Z_i(x_1, x_2, \ldots, x_i) - Z_i(x_1, x_2, \ldots, x_i')| = \left| \frac{\sum_{y_{i+1}, \ldots, y_N} (Z(\omega) - Z(f(\omega)))}{(N-i)!} \right| \leq 1.$$

□