Hamilton cycles in random graphs of minimal degree at least k

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0 Introduction

This paper is a contribution to the theory of random graphs (see Bollobás [4]). Thanks to the efforts of a great many people, including Pósa [13], Korshunov [12], Komlós and Szemerédi [11] and Bollobás [2], the threshold function of a Hamilton cycle is known to be $\frac{1}{2}n(\log n + \log\log n + \omega(n))$, where $\omega(n) \to \infty$.

In fact, Bollobás [3] showed that almost every random graph process is such that the hitting time of minimal degree 2 is equal to the hitting time of a Hamilton cycle. Thus if at time 0 we start with the empty graph with vertex set $[n] = \{1, 2, \ldots, n\}$ and at time t, $1 \leqslant t \leqslant N = \binom{n}{2}$, we add the t-th edge at random, then in almost every case it is true that if we stop as soon as the minimal degree becomes 2, the graph at hand is Hamiltonian. Since, trivially, a graph of minimal degree less than 2 is not Hamiltonian, this means that the primary obstruction to a Hamilton cycle is the existence of a vertex of degree less than 2. It is a classical result of Erdős and Rényi [7] that the threshold function of the minimal degree being at least 2 is $\frac{1}{2}n(\log n + \log\log n + \omega(n))$, where $\omega(n) \to \infty$, so, in particular, the threshold function of a Hamilton cycle is also $\frac{1}{2}n(\log n + \log\log n + \omega(n))$.

A simpler result in the vein of the results above, proved by Erdős and Rényi [8], is that the primary obstruction to a matching (assuming that n, the number of vertices, is even) is the existence of an isolated vertex. In particular, having a matching has the same threshold function as having minimal degree 1, namely $\frac{1}{2}n(\log n + \omega(n))$, where $\omega(n) \to \infty$.

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Bollobás & Frieze [6] studied the secondary obstructions to a matching in a graph of even order. If we have a condition on the minimal degree being at least 1 then the threshold function goes down to about half the original one: $\frac{1}{2}n(\frac{1}{2}\log n + \log\log n + \omega(n))$, where $\omega(n) \to \infty$, the main secondary obstruction being the existence of two vertices of degree 1 having a common neighbour.

Our main aim in this paper is the study of secondary obstructions to a Hamilton cycle. If we condition on the minimal degree being at least 2 then what is the threshold function of a Hamilton cycle and what is the crucial obstruction to a Hamilton cycle? We shall show that the secondary obstruction is a 2-*spider*: three vertices of degree 2 having a common neighbour; the new threshold function is

$$\tfrac{1}{2}n(\tfrac{1}{3}\log n + 2\log\log n + \omega(n)), \qquad \text{where } \omega(n) \to \infty.$$

In fact, we shall study the secondary obstructions to $\lfloor \frac{1}{2}k \rfloor$ edge-disjoint Hamilton cycles and $k - 2\lfloor \frac{1}{2}k \rfloor$ matchings, the obvious primary obstruction being the existence of a vertex of degree less than k. These turn out to be k-*spiders*: $k+1$ vertices of degree k having a common neighbour.

1 Generating a random graph of minimal degree at least k

For a detailed study of the standard models of random graphs we refer the reader to [4]. Here we shall concern ourselves with a natural model which has rarely been studied because of the technical difficulties involved. Given natural numbers n, m and k, let $\mathcal{G}(n, m; \delta \geqslant k)$ be the set of all graphs with vertex set $[n] = \{1, 2, \ldots, n\}$ having m edges and minimal degree at least k. Let us turn $\mathcal{G}(n, m; \delta \geqslant k)$ into a probability space by giving all members of it the same probability. In this paper we shall study this probability space, but as the space itself is not amenable to direct investigation, we shall generate the members of $\mathcal{G}(n, m; \delta \geqslant k)$ in a rather roundabout way.

We shall need a probability space rather close to $\mathcal{G}(n, m; \delta \geqslant k)$: the space $\mathcal{MG}(n, m; \delta \geqslant k)$ consisting of all multigraphs on $[n]$ with m edges and loops, having minimal degree at least k, with all members of this set equiprobable.

For natural numbers s and t, let $[s]^t$ be the set of all s^t sequences of length t with the terms taken from the set $[s] = \{1, 2, \ldots, s\}$. Consider $[s]^t$ as a probability space in which any two points (i.e., sequences) have the same probability, namely s^{-t}. The space $[s]^t$ has the following intuitive interpretation which we shall use in the sequel. Put t distinguishable balls, say $b_1, b_2, \dots, b_t$ into s boxes, with probability $1/s$ of putting a ball into any of the boxes. Every arrangement corresponds to a sequence of length t: if b_j goes into the i-th box then set $x_j = i$. Then the sequence $(x_1, x_2, \dots, x_t)$ is a random element of the space $[s]^t$.

The *degree* of a number i in a sequence $X = (x_1, x_2, \dots, x_t) \in [s]^t$, denoted by $d_X(i)$, is the number of times i occurs in the sequence: $d_X(i) = |\{j : x_j = i\}|$. Thus $d_X(i)$ is the number of balls in the i-th box. The *minimal degree* of X is $\delta(X) = \min\{d_X(i) : i \in [s]\}$. Similarly the *maximal degree* of X is $\Delta(X) = \max\{d_X(i) : i \in [s]\}$.

Let $[n \mid \delta \geqslant k]^l = \{X \in [n]^l : \delta(X) \geqslant k\}$ and consider this set as a probability space consisting of equiprobable elements. This space is much less pleasant than $[s]^t$ but it is not very far from $\mathcal{G}(n, m; \delta \geqslant k)$, the probability space we intend to study. Indeed for $Y = (y_1, y_2, \dots, y_l) \in [n \mid \delta \geqslant k]^l$ (l even) we take the multigraph with vertex set $[n]$ and edge set $\{y_1y_2, y_3y_4, \dots, y_{l-1}y_l\}$. Ignoring the loops and replacing multiple edges by simple edges, we obtain a graph with vertex set $[n]$. Conditional on this graph having precisely m edges and minimal degree at least k, as we shall see, we obtain exactly a random element of $\mathcal{G}(n, m; \delta \geqslant k)$.

Let us see then how we can pass from $[s]^t$ to $[n \mid \delta \geqslant k]^l$. For $X \in [s]^t$, let

$$U(X) = U(X, k) = \{i \in [s] : d_X(i) \geqslant k\}$$
$$= \{i_1, i_2, \dots, i_n\},$$

where $i_1 < i_2 < \cdots < i_n$. Omit the terms of X not belonging to U and replace i_r by r. Let $\rho(X)$ be the sequence obtained in this way; call $\rho(X)$ the *reduced sequence*. By construction, $\rho(X) \in [n \mid \delta \geqslant k]^l$ for some l. We call n the *order* and l the *length* of the reduced sequence. For example, if $s=7$, $t = 16$ and $k = 2$ then from

$$X = (4, 7, 1, 5, 6, 7, 2, 7, 1, 4, 7, 2, 3, 5, 7, 4)$$

we first obtain

$$(4, 7, 1, 5, 7, 2, 7, 1, 4, 7, 2, 5, 7, 4),$$

as $U(X) = \{1, 2, 4, 5, 7\}$, and then

$$\rho(X) = (3, 5, 1, 4, 5, 2, 5, 1, 3, 5, 2, 4, 5, 3) \in [5 \mid \delta \geqslant 2]^{14}.$$

Let us use the reduced sequence $\rho(X) = (y_1, y_2, \dots, y_l) \in [n \mid \delta \geqslant k]^l$ to construct a multigraph $MG(X, k)$ as follows: the vertex set is $[n]$ and the edge set is $\{y_1y_2, y_3y_4, \ldots, y_{l'-1}y_{l'}\}$, where $l' = 2\lfloor\frac{1}{2}l\rfloor$. Finally, let $G(X,k)$ be the graph obtained from $MG(X,k)$ by deleting the loops and replacing the multiple edges by simple edges. Let us write $v(X)$ and $l(X)$ for the order and length of $\rho(X)$, let $e_m(X) = \lfloor\frac{1}{2}l(X)\rfloor$ be the number of edges and loops of $MG(X,k)$ and let $e(X)$ be the number of edges of $G(X,k)$. Note that the number of vertices of $G(X,k)$ (and $MG(X,k)$) is also $v(X)$.

For $Y \in [n \mid \delta \geqslant k]^l$ write $P(\rho(X) = Y) = P(X \in [s]^t : \rho(X) = Y)$, with the probability taken in $[s]^t$.

Lemma 1 *Let* $Y_1, Y_2 \in [n \mid \delta \geqslant k]^l$. *Then*

$$P(\rho(X) = Y_1) = P(\rho(X) = Y_2).$$

Proof Let $Y \in [n \mid \delta \geqslant k]^l$. In how many ways can we choose a sequence $X \in [s]^t$ satisfying $\rho(X) = Y$? The set $U(X)$ can be chosen in $\binom{s}{n}$ ways and the set $W(X) = \{j : x_j \in U\}$ can be chosen in $\binom{t}{l}$ ways. Having chosen $U(X)$ and $W(X)$, we have fixed l terms of the sequence X. The remaining $t-l$ terms come from a set of $s-n$ elements, with no element occuring more than $k-1$ times. Hence $P(X \in [s]^t : \rho(X) = Y)$ does not depend on Y, provided $Y \in [n \mid \delta \geqslant k]^l$. □

Lemma 1 states that conditional on $v(X) = n$ and $l(X) = l$, all sequences in $[n \mid \delta \geqslant k]^l$ are equally likely to arise as $\rho(X)$. A similar assertion holds for $G(X,k)$ and $\mathcal{G}(n,m;\delta \geqslant k)$.

Lemma 2 *If* $G_1, G_2 \in \mathcal{G}(n,m;\delta \geqslant k)$ *then*

$$P(G(X,k) = G_1) = P(G(X,k) = G_2),$$

with the probabilities taken in $[s]^t$.

Proof Fix an ordering of $[n]^{(2)}$, the set of possible edges, say take the lex order: 12, 13, …, $1n$, 23, 24, …, $2n$, 34, …, $(n-1)n$.

Pick two sequences of non-negative integers $(\mu_j)_1^m$ and $(\nu_i)_1^n$, such that $\mu_j \geqslant 1$ for every j and $m_1 = \mu + \nu \leqslant \frac{1}{2}t$, where $\mu = \sum \mu_j$ and $\nu = \sum \nu_i$.

Given a graph $G \in \mathcal{G}(n,m;\delta \geqslant k)$, let $\varphi(G)$ be the multigraph obtained from G by taking the j-th edge with multiplicity μ_j and adding ν_i loops at vertex i. All we have to check is that the number of sequences X for which $MG(X,k) = \varphi(G)$ is independent of G.

Suppose that $MG(X,k) = \varphi(G)$. The multigraph $\varphi(G)$ has m_1 edges and loops, so the set of edges and loops can be ordered in

$$\binom{m_1}{\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n}$$

ways; furthermore, the edges can be oriented in 2^μ ways. The list of oriented edges and unoriented loops determines the first $2m_1$ terms of the reduced sequence $\rho(X)$. If $\rho(X)$ has more than $2m_1$ terms then it has precisely $2m_1+1$ terms, giving us n choices for the last term. By Lemma 1, this shows that the number of sequences $X \in [s]^t$ for which $MG(X,k) = \varphi(G)$ is independent of G. □

It is easily seen that Lemma 2 does not extend to $MG(X,k)$ and $\mathcal{MG}(n,m;\delta \geq k)$. However, the proof of Lemma 2 implies a variant of it for multigraphs with given sequences $(\mu_j)_1^m$ and $(\nu_i)_1^n$.

Our main problem in investigating the space $\mathcal{G}(n,m;\delta \geq k)$ is that, although for a given space $[s]^t$ there is a pair (n,m) such that $P(v(X) = n$ and $e(X) = m)$ is rather large (in the sense that it is much larger than the average though still close to 0), we cannot pinpoint the exact dependence of a pair (n,m) on (s,t). As we cannot come close to determining this exact dependence, we cannot find suitable pairs $(n,m) = (n,m(n))$ $(n = 1,2,\ldots,)$ which are reached with not too small a probability from a pair $(s,t) = (s(n),t(n))$. To overcome this difficulty, we shall nest the original models $[s]^t$ in such a way that a small change in the parameters (s,t) yields only a small change in the distribution of $(v(X),e(X))$. Then we shall locate rather crudely the peak of this distribution and shall show that if (s,t) is varied suitably then for some choice of (s,t) we must hit our preselected pair (n,m) with a reasonable probability.

Let Σ_s be the set of all $s!$ sequences of natural numbers $\boldsymbol{a} = (a_1,a_2,\ldots,a_s)$ of length s such that $a_i \leq i$ for every i. Consider Σ_s as a probability space by giving all sequences the same probability, $1/s!$. Given $\boldsymbol{a} \in \Sigma_s$ define $\beta(\boldsymbol{a}) = (b_1,\ldots,b_s)$ by setting $b_1 = a_1 = 1$ and, having defined $b_1,\ldots,b_i$ for $i < s$,

$$b_{i+1} = \begin{cases} b_i & \text{if } a_{i+1} \leq i, \\ i+1 & \text{if } a_{i+1} = i+1. \end{cases}$$

It is easily checked that if $1 \leq h \leq i$ then

$$P(b_i = h) = P(b_i = h \mid b_{i+1} = i+1,\, b_{i+2} = c_{i+2},\ldots,\, b_s = c_s) = \frac{1}{i} \quad (1)$$

and

$$P(b_i = b_{i-1} \mid b_i \leq i-1) = 1. \quad (2)$$

To see this, note that $P(b_i = i) = P(a_i = i) = 1/i$ and, by induction on

i, for $h \leqslant i-1$ we have

$$P(b_i = h) = P(a_i \leqslant i-1)P(b_{i-1} = h) = \frac{i-1}{i}\frac{1}{i-1} = \frac{1}{i}.$$

The probability space $\Sigma_s^t = \Sigma_s \times \cdots \times \Sigma_s$ allows us to pick a random sequence from each member of a family of probability spaces $[i]^j$; this will enable us to pass from a random sequence in a space $[i]^j$ to a random sequence in a space $[i_0]^{j_0}$. Indeed, for

$$A = (a^{(1)}, a^{(2)}, \ldots, a^{(t)}) \in \Sigma_s^t, \qquad 1 \leqslant i \leqslant s, \qquad 1 \leqslant j \leqslant t,$$

define

$$X(i,j) = X_A(i,j) = \big(x(i,1), x(i,2), \ldots, x(i,j)\big),$$

where $x(i,l)$ is defined by

$$\beta(a^{(l)}) = \big(x(1,l), x(2,l), \ldots, x(s,l)\big).$$

Since, by relation (1), for fixed i and l, the random variable $x(i,l)$ is uniformly distributed on $[i]$, we have the following lemma.

Lemma 3 *Let $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$. Then the map $\Sigma_s^t \to [i]^j$, given by $A \mapsto X_A(i,j)$, is a measure-preserving map onto $[i]^j$.*

There is another, more intuitive, way of describing a random array $(X_A(i,j))$. This time we start with the only element of $[1]^t$, namely $(1,1,\ldots,1)$, we change it randomly into an element of $[2]^t$, then apply another random transformation to obtain an element of $[3]^t$, and so on. What are these random transformations we apply? Having got a random element $X(i,t)$ of $[i]^t$, say $\big(x(i,1), x(i,2), \ldots, x(i,t)\big)$, let

$$x(i+1,j) = \begin{cases} x(i,j) & \text{with probability } \dfrac{i}{i+1}, \\[2ex] i+1 & \text{with probability } \dfrac{1}{i+1}. \end{cases}$$

Then $X(i+1,t) = \big(x(i+1,1), x(i+1,2), \ldots, x(i+1,t)\big)$ is our random element of $[i+1]^t$. For $1 \leqslant j \leqslant t$ we take

$$X(i,j) = \big(x(i,1), x(i,2), \ldots, x(i,j)\big).$$

Relations (1) and (2) imply that the random array $\{X(i,j) : 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant t\}$ obtained in this way has the same distribution as the array $\{X_A(i,j) : 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant t\}$, with A chosen from Σ_s^t.

Lemma 4 *Let $A = (a^{(1)}, a^{(2)}, \ldots, a^{(t)}) \in \Sigma_s^t$. Then the random variables $\beta(a^{(j)}) = \big(x(1,j), x(2,j), \ldots, x(s,j)\big)$, $1 \leqslant j \leqslant t$, are independent and,* *conditional on* $x(1,j), x(2,j), \ldots, x(i-1,j)$,

$$x(i,j) = \begin{cases} i & \text{with probability } \dfrac{1}{i}, \\ x(i-1,j) & \text{with probability } 1-\dfrac{1}{i}. \end{cases}$$

Let now $1 \leqslant m \leqslant N$ be a natural number satisfying

$$m = m(n) = \tfrac{1}{2}n\left(\frac{1}{k+1}\log n + k\log\log n + c\right) + O(1),$$

where k is a fixed natural number and c is a real constant. Furthermore, set

$$s_0 = n, \qquad s = n + n^{k/(k+1)},$$
$$t_0 = 2m, \qquad t = 2m + n^{k/(k+1)}.$$

Strictly speaking, we should set

$$s = n + \lfloor n^{k/(k+1)} \rfloor \qquad \text{and} \qquad t = 2m + \lfloor n^{k/(k+1)} \rfloor$$

but, in order to avoid inessential, purely formal complications, throughout this paper we shall dispense with the integer signs. In addition, all our inequalities are claimed to hold if n is *sufficiently large*.

Given a sequence $X = (x_1, x_2, \ldots, x_j) \in [i]^j$, set

$$D_l(X) = |\{h : d_X(h) = l\}|,$$

that is, denote by $D_l(X)$ the number of boxes containing precisely l balls.

Lemma 5 *Set*

$$D_l = \frac{1}{\min\{k!, l!\}(k+1)^l} e^{-c} (\log n)^{l-k} n^{k/(k+1)} \left(1 + \frac{1}{\sqrt{\log n}}\right), \qquad (3)$$

and let $E_0 \subset \Sigma_s^t$ *be the event*

$$\{A \in \Sigma_s^t : D_l(X_A(i,j)) \leqslant D_l \text{ for all } 0 \leqslant l \leqslant t,\ s_0 \leqslant i \leqslant s \text{ and } t_0 \leqslant j \leqslant t\}.$$

Then

$$P(\overline{E}_0) = o(n^{-\log n}).$$

Proof The bound above is tighter than needed immediately, but is important later. Fix i and j ($s_0 \leqslant i \leqslant s$, $t_0 \leqslant j \leqslant t$) and, for simplicity, write $d = D_l$ and let $\lambda = (\log n)^3$. The (generalized) Markov inequality implies

$$
\begin{aligned}
P[D_l(X_A(i,j)) > d]& \\
&\leq \binom{d}{\lambda}^{-1} E\left[\binom{D_l(X_A(i,j))}{\lambda}\right] \\
&\leq \binom{d}{\lambda}^{-1}\binom{i}{\lambda}\frac{j^{\lambda l}}{(l!)^{\lambda}}\left(\frac{1}{i}\right)^{\lambda l}\left(1-\frac{\lambda}{i}\right)^{j-\lambda l} \\
&\leq \left[\frac{i-\lambda}{d-\lambda}\left(\frac{j}{i}\right)^{l}\frac{1}{l!}\exp\left(-\frac{j}{i}+\frac{\lambda l}{i}\right)\right]^{\lambda} \\
&= \left[\frac{1}{d}\frac{e^{-c}}{(k+1)^{l}}(\log n)^{l-k}n^{k/(k+1)}\frac{1}{l!}\left\{1+O\left(\frac{\log\log n}{\log n}\right)\right\}^{l}\right]^{\lambda} \\
&\leq \left(1+\frac{1}{2\sqrt{\log n}}\right)^{-\lambda}.
\end{aligned}
$$

Since we have fewer than n^4 choices for the triple (l,i,j), the assertion follows. □

Lemma 6 *Almost every $A \in \Sigma_s^t$ is such that*

$$e_m(X_A(i,j)) - e(X_A(i,j)) \leq 8(\log n)^4$$

for all i, j, $s_0 \leq i \leq s$ and $t_0 \leq j \leq t$.

The proof of Lemma 6 is deferred to the next section as it relies upon concepts developed in that section.

Lemmas 5 and 6 enable us to show that as we vary i and j, the functions $v(X_A(i,j))$, $e(X_A(i,j))$ and $e_m(X_A(i,j))$ vary in a more or less predictable way.

Lemma 7 *Let the maps $\Sigma_s^t \to [i]^j$, given by $A \mapsto X(i,j) = X_A(i,j)$, be as before. Then almost every A is such that if $s_0 \leq i \leq s$ and $t_0 \leq j \leq t$ then*

$$v(X(s_0,j)) \leq n \leq v(X(s,j)), \tag{4}$$

$$e_m(X(i,t_0)) \leq m \leq e(X(i,t)), \tag{5}$$

$$|f(X(i,j+1)) - f(X(i,j))| < 9(\log n)^4, \tag{6}$$

$$|f(X(i+1,j)) - f(X(i,j))| < 9(\log n)^4, \tag{7}$$

for $f = v$, e and e_m.

Proof Let $E_1 \subset \Sigma_s^t$ be the event that E_0 (see Lemma 5) holds and so does the conclusion of Lemma 6. We have to show that the probability of $A \in E_1$ not satisfying all the inequalities (4), (5), (6) and (7) tends to 0 as $n \to \infty$.

Note that

$$v(X(i,j)) = i - \sum_{l=0}^{k-1} D_l(X(i,j)) \tag{8}$$

and

$$e_m(X(i,j)) = \left\lfloor \frac{1}{2}\left(j - \sum_{l=1}^{k-1} lD_l(X(i,j))\right)\right\rfloor. \tag{9}$$

Since for $A \in E_1$ we have

$$0 \leqslant D_l(X(i,j)) \leqslant D_l \leqslant \frac{2e^{-c}}{(k+1)^l}(\log n)^{l-k} n^{k/(k+1)} \tag{10}$$

and

$$0 \leqslant e_m(X(i,j)) - e(X(i,j)) \leqslant 8(\log n)^4 \tag{11}$$

for all i and j ($s_0 \leqslant i \leqslant s$, $t_0 \leqslant j \leqslant t$), equations (8) and (9) imply (4) and (5).

Let us turn to the proof of (6) and (7). When we add the $(j+1)$-th ball, there is precisely one box whose contents changes, namely it has one more ball. Hence

$$0 \leqslant v(X(i,j+1)) - v(X(i,j)) \leqslant 1 \tag{12}$$

and

$$0 \leqslant e_m(X(i,j+1)) - e_m(X(i,j)) \leqslant \tfrac{1}{2}(k+1). \tag{13}$$

Inequalities (11), (12) and (13) imply (6).

In order to prove some analogous inequalities about $X(i+1,j)$ and $X(i,j)$, we shall make use of Lemma 4. Given

$$\sum_{l=0}^{k+1} lD_l = O((\log n) n^{k/(k+1)})$$

elements of $X(i,j)$, the probability that when passing to $X(i+1,j)$ we change at least $2k+1$ of them (into $i+1$) is

$$O((n^{k/(k+1)} \log n)^{2k+1})n^{-2k-1} = o(n^{-2k/(k+1)}).$$

Since for $s_0 \leqslant i \leqslant s$ and $t_0 \leqslant j \leqslant t$ we have about $n^{2k/(k+1)}$ choices for (i,j), almost every $A \in E_1$ is such that at most $2k$ of the balls belonging to boxes with at most $k+1$ balls are changed at each stage $X_A(i,j) \to X_A(i+1,j)$.

Similarly, the probability that, given $D_{k+l} = O((\log n)^l n^{k/(k+1)})$ groups of $k+l \geqslant k+2$ balls each, we change at least $l+1$ in some group is at most

$$D_{k+l}\binom{k+l}{l+1}n^{-l-1} = O((\log n)^l n^{k/(k+1)} n^{-l-1})$$
$$= O\left(n^{-1/(k+1)}\left\{\frac{\log n}{n}\right\}^l\right).$$

Hence, summing these inequalities for $l \geq 2$, we see that almost every $A \in E_1$ is such that, when changing $X_A(i,j)$ into $X_A(i+1,j)$, no box with at least $k+2$ balls is reduced to a box with fewer than k balls.

Consequently, almost every $A \in E_1$ is such that

$$0 \leq \sum_{l=0}^{k-1} D_l(X(i+1,j)) - \sum_{l=0}^{k-1} D_l(X(i,j)) \leq 2k+1$$

and

$$-2k \leq \sum_{l=0}^{k-1} lD_l(X(i+1,j)) - \sum_{l=0}^{k-1} lD_l(X(i,j)) \leq 2k(k-1)+k-1 \leq 2k^2.$$

The additional terms 1 and $k-1$ arise because it may happen that the $(i+1)$-th box contains at most $k-1$ balls. Hence

$$-2k \leq v(X(i+1,j)) - v(X(i,j)) \leq 1 \tag{14}$$

and

$$-k^2 \leq e_m(X(i+1,j)) - e_m(X(i,j)) \leq k. \tag{15}$$

Inequalities (11), (14) and (15) imply inequality (7). □

Lemma 8 *Let n, m, s_0, s, t_0 and t be as before and set $L = L(n) = 9(\log n)^4$.*

(a) *There exist i, j, n' and m' with $|n-n'| \leq L$, $|m-m'| \leq L$, $s_0 \leq i \leq s$ and $t_0 \leq j \leq t$ such that*

$$P[v(X(i,j)) = n' \text{ and } e_m(X(i,j)) = m'] > 10^{-3}(\log n)^{-8} n^{-2k/(k+1)}.$$

(b) *There exist i, j, n'' and m'' with $|n-n''| \leq L$, $|m-m''| \leq L$, $s_0 \leq i \leq s$ and $t_0 \leq j \leq t$ such that*

$$P[v(X(i,j)) = n'' \text{ and } e(X(i,j)) = m''] > 10^{-3}(\log n)^{-8} n^{-2k/(k+1)}.$$

Proof (a) Let $E_2 \subset \Sigma_s^t$ be the set of A's satisfying the conclusions of Lemma 7 so that, by Lemma 7, $P(E_2) = 1-o(1)$. Let

$$\Lambda = \{(i,j) \in \mathbb{Z}^2 : s_0 \leq i \leq s,\ t_0 \leq j \leq t\} \subset Q,$$

where

$$Q = \{(i,j) \in \mathbb{R}^2 : s_0 \leq i \leq s,\ t_0 \leq j \leq t\} \subset \mathbb{R}^2.$$

Define $F: \Lambda \to \mathbb{R}^2$ by

$$F(i,j) = \big(v(X(i,j)), e_m(X(i,j))\big).$$

By inequalities (6) and (7), this function F can be extended to a function $\tilde{F}: Q \to \mathbb{R}^2$ such that

$$\|\tilde{F}(x',y') - \tilde{F}(x,y)\|_\infty \leq 2L \max\{|x-x'|, |y-y'|\}$$

and $\tilde{F}$ is linear on the segments $[(i,j),(i+1,j)]$ and $[(i,j),(i,j+1)]$.

We claim that $\tilde{F}$ maps some point $(x,y) \in Q$ into (n,m). Indeed, by inequalities (4) and (5), the function $\tilde{F}$ maps the side $x = s_0$ of Q into the half-plane $\{(x,y) : x \leq n\}$, the side $y = t$ into the half-plane $\{(x,y) : y \geq m\}$, $x = s$ into $\{(x,y) : x \geq n\}$, and $y = t_0$ into $\{(x,y) : y \leq m\}$. Hence the image of the boundary of Q has winding number 1 about (n,m), unless (n,m) is in the image of the boundary. But this implies that the continuous function $\tilde{F}$ maps some point of Q into (n,m).

Having found $(x,y) \in Q$ with $\tilde{F}(x,y) = (n,m)$, let $(i,j) \in \Lambda$ be a lattice point satisfying $\max\{|i-x|, |j-y|\} \leq \frac{1}{2}$. Then $\tilde{F}(i,j) = F(i,j)$ is close enough to (n,m):

$$|v(X(i,j)) - n| \leq L \quad \text{and} \quad |e_m(X(i,j)) - m| \leq L. \tag{16}$$

Since for every $A \in E_2$ there is a point $(i,j) \in \Lambda$ satisfying (16) and we have $|\Lambda| \sim n^{2k/(k+1)}$ choices for (i,j), some point $(i,j) \in \Lambda$ will do for at least $\frac{2}{3}n^{-2k/(k+1)}$ portion of E_2. As, moreover, there are $(2L+1)^2$ pairs (v, e_m) satisfying (16), the assertion (a) follows.

The proof of (b) is analogous. □

We are ready to prove the main theorem of the section about generating random graphs and multigraphs of minimal degree at least k.

Theorem 9 *Let*

$$m = m(n) = \tfrac{1}{2}n\left(\frac{1}{k+1}\log n + k \log\log n + c\right),$$

where k is a natural number and $c \in \mathbb{R}$.

(a) *There exist* i' *and* j' *such that* $n \leq i' \leq n + n^{k/(k+1)}$, $2m \leq j' \leq 2m + n^{k/(k+1)}$ *and*

$$P[v(X(i',j')) = n \text{ and } e_m(X(i',j')) = m] \geq 10^{-4}(\log n)^{-8} n^{-2k/(k+1)}.$$

(b) *There exist* i'' *and* j'' *such that* $n \leq i'' \leq n + n^{k/(k+1)}$, $2m \leq j'' \leq 2m + n^{k/(k+1)}$ *and*

$$P[v(X(i'',j'')) = n \text{ and } e(X(i'',j'')) = m] \geq 10^{-4}(\log n)^{-8} n^{-2k/(k+1)}.$$

Proof (a) Let i, j, n' and m' be as in Lemma 8 (a): with probability at least $10^{-3}(\log n)^{-8}n^{-2k/(k+1)}$ the pair $(v(X(i,j)), e_m(X(i,j))) = (n',m')$ is close to (n,m), so let $a = n-n'$ and $b = m-m'$.

We claim that $i' = i+a$ and $j' = j+2b$ will do for (a). To see this, start with $MG(X_A(i,j),k)$ and add or remove $|a|$ boxes and $2|b|$ balls to $X_A(i,j)$, as appropriate, that is, consider the multigraph $MG(X_A(i+a,j+2b),k)$. Let E_0 be the event defined in Lemma 5. Recalling the calculations in the proof of Lemma 7, we see that for $A \in E_0$ the probability that $MG(X_A(i+1,j),k)$ has one more vertex than $MG(X_A(i,j),k)$ and these multigraphs have the same number of edges is at least

$$1-O(n^{-1/(k+1)})-O(n^{-(k+2)/(k+1)}\log n) = 1-O(n^{-1/(k+1)}), \quad (17)$$

the second term being the probability that we change a ball in a box with at most k balls, or that the box $i+1$ contains fewer than k balls, and the third is the probability that we change at least $l+1$ balls in a box with $k+l$ balls ($l \geqslant 1$).

Similarly, for $A \in E_0$ the probability that $MG(X(i,j+2),k)$ has precisely one more edge than $MG(X(i,j),k)$ and they have the same number of vertices is at least

$$(1-n^{k/(k+1)}n^{-1})^2 \geqslant 1-2n^{-1/(k+1)}, \quad (18)$$

where $n^{k/(k+1)}n^{-1}$ is an upper bound for the probability that a ball is added to a box containing at most $k-1$ balls.

Since $a+b = o(n^{1/(k+1)})$, part (a) of the theorem follows, recalling that by (4) and (5) the numbers i' and j' satisfy the required inequalities, and noting that by changing i to $i-1$ and j to $j-2$, if necessary, the result holds whatever the signs of a and b.

(b) This is proved analogously. For the case of adding the $(i+1)$-th box, we have to subtract a term of order $(\log n)^2/n$ from the left-hand side of (17) to account for the probability that either a loop or multiple edge has at least one of its end vertices changed to vertex $i+1$, or that $MG(X(i+1,j),k)$ has some loop or multiple edge incident with vertex $i+1$.

For the case of adding balls $j+1$ and $j+2$, we have to subtract a term of order $\log n/n$ from the left-hand side of (18) to account for the probability that the 2 new balls yield a loop or a multiple edge.

We also need $A \in E_1$ rather than E_0 for (5) to hold (see Lemma 7). □

2 Configurations

We describe a useful way of partitioning $[n \mid \delta \geqslant k]^l$ according to the degrees of the numbers in the sequence.

Let

$$DX(k,\nu,\mu) = \left\{ \boldsymbol{d} \in [\mu]^\nu : \sum_{i=1}^{\nu} d_i = \mu \text{ and } d_i \geqslant k \text{ for } i = 1,2,\ldots,\nu \right\},$$

where μ is even. For $\boldsymbol{d} \in DX(k,\nu,\mu)$ let

$$[\nu|\boldsymbol{d}]^\mu = \{X \in [\nu]^\mu : d_X(i) = d_i \text{ for } i = 1,2,\ldots,\nu\}.$$

It turns out that several properties can most easily be proved by conditioning on $\rho(X) \in [\nu|\boldsymbol{d}]^\mu$ for fixed $\boldsymbol{d} \in DX(k,\nu,\mu)$ for some ν and μ.

We work with $MG(X,k)$ conditioned on $\rho(X) \in [\nu|\boldsymbol{d}]^\mu$. This has the same distribution as the multigraph produced in the configuration model of Bollobás [4]. Thus let $W_1, W_2, \ldots, W_\nu$ be disjoint sets, where $|W_i| = d_i$ for $i = 1,2,\ldots,\nu$, and let $W = \bigcup_{i=1}^{\nu} W_i$. For $S \subset [\nu]$ let $W_S = \bigcup_{i \in S} W_i$ and for $w \in W$ we define $\psi(w)$ by $w \in W_{\psi(w)}$. A configuration F is a partition of W into $\frac{1}{2}\mu$ pairs. Φ is the set of configurations and for $F \in \Phi$ we let $\varphi(F)$ be the multigraph with vertices $[\nu]$ and an edge $\{\psi(x),\psi(y)\}$ for each pair $\{x,y\} \in F$.

We claim that if each $F \in \Phi$ is equally likely then $\varphi(F)$ is distributed as $MG(X,k)$ conditional on $\rho(X) \in [\nu|\boldsymbol{d}]^\mu$. To see this, note the following.

We can generate F at random by taking a random permutation $w_1, w_2, \ldots, w_\mu$ of W and then taking pairs $\{w_1,w_2\}$, $\{w_3,w_4\}$, ... (Note that each F appears $(\frac{1}{2}\mu)!\,2^{\frac{1}{2}\mu}$ times). (19a)

If we replace each w_i by $\psi(w_i)$ then we obtain a member of $[\nu|\boldsymbol{d}]^\mu$ and each such member appears $\prod_{i=1}^{\nu} d_i!$ times. (19b)

The next lemma gives a list of the likely properties of the sequence $\boldsymbol{d}_{\rho(X)}$.

Lemma 10 *Let s_0, s, t_0 and t be as in the previous section and suppose that i and j satisfy $s_0 \leqslant i \leqslant s$ and $t_0 \leqslant j \leqslant t$.*

Let $DX_0 = \{\boldsymbol{d} \in \bigcup_{\nu,\mu} DX(k,\nu,\mu) : (20) \text{ below holds}\}$. Then

$$P(\boldsymbol{d}_{\rho(X)} \notin DX_0) = o(n^{-\log n}),$$

where $X = X(i,j)$.

$$d_r \leqslant (\log n)^2 \qquad (r = 1,2,\ldots,\nu). \tag{20a}$$

$$\left| |\{r : d_r = k\}| - \frac{e^{-c}}{k!\,(k+1)^k} n^{k/(k+1)} \right| = o(n^{k/(k+1)}). \quad \text{(20b)}$$

$$\left| \left\{ r : d_r < (1-\epsilon_n)\frac{\log n}{k+1} \text{ or } d_r > (1+\epsilon_n)\frac{\log n}{k+1} \right\} \right| < 2n^{1-\epsilon_n^2/(4k+4)}, \quad \text{(20c)}$$

where $\epsilon_n = 1/\log\log n$.

$$|\text{TINY}| \leqslant n^{(2k+1)/(2k+3)}, \quad \text{(20d)}$$

where

$$\text{TINY} = \left\{ r : d_r \leqslant \frac{\alpha_k \log n}{k+1} + 2 \right\}$$

and where α_k *satisfies*

$$\frac{\alpha_k}{k+1} \log \frac{e}{\alpha_k} = \frac{1}{2(k+1)(k+2)}.$$

(The exact value for α_k is unimportant; we only require that α_k is positive and sufficiently small.)

Proof (a) Now $\Delta(\rho(X)) = \Delta(X)$ and

$$\begin{aligned} P(\Delta(X) \geqslant (\log n)^2) &\leqslant iP(d_X(1) \geqslant (\log n)^2) \\ &\leqslant i\binom{j}{(\log n)^2}\left(\frac{1}{i}\right)^{(\log n)^2} \\ &= o(n^{-\log n}). \end{aligned} \quad \text{(21)}$$

(b) Lemma 5 gives an upper bound to the number of r such that $d_X(r) = k$ which is tight enough and is satisfied with the required probability. Now let $Z_k = |\{r : d_X(r) \geqslant k+1\}|$, $\lambda = n^{1/(k+1)}(\log n)^3$ and

$$K = \frac{e^{-c}}{k!\,(k+1)^k} n^{-1/(k+1)}.$$

Then, using a simple monotonicity argument for the third inequality,

$$\begin{aligned} P\left(Z_k \geqslant \hat{\imath} = i - \left[1 - \frac{1}{\sqrt{\log n}}\right] \frac{e^{-c}}{k!\,(k+1)^k} n^{k/(k+1)} \right) &\leqslant \binom{\hat{\imath}}{\lambda}^{-1} E\left[\binom{Z_k}{\lambda}\right] \\ &\leqslant \binom{\hat{\imath}}{\lambda}^{-1}\binom{i}{\lambda} P(d_X(r) \geqslant k+1,\ r = 1,2,\ldots,\lambda) \\ &\leqslant \binom{\hat{\imath}}{\lambda}^{-1}\binom{i}{\lambda} P(d_X(1) \geqslant k+1)^{\lambda} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\frac{i-\lambda}{\hat{\imath}-\lambda}\left[1-K\left\{1-O\left(\frac{\log\log n}{\log n}\right)\right\}\right]\right)^{\lambda} \\
&\leqslant \left(\left[1+\left(1-\frac{1}{2\sqrt{\log n}}\right)K\right]\left[1-K\left\{1-O\left(\frac{\log\log n}{\log n}\right)\right\}\right]\right)^{\lambda} \\
&\leqslant \left(1-\frac{K}{3\sqrt{\log n}}\right)^{\lambda} \\
&= o(n^{-\log n}).
\end{aligned}
$$

(c) In the proof of (c) we need an inequality for the binomial random variable $B(n,p)$ (see [4, p.13, Theorem 7 (*i*)]): if $0 < p \leqslant \frac{1}{2}$, $0 < \delta \leqslant \frac{1}{12}$ and $\delta^2 pqn \geqslant 1$ then

$$
P(|B(n,p)-pn| \geqslant \delta pn) \leqslant e^{-\delta^2 np/3}. \tag{22}
$$

Let now

$$
Z_{>} = \left|\left\{r : d_X(r) \geqslant \frac{1+\epsilon_n}{k+1}\log n\right\}\right|, \qquad \lambda = (\log n)^2.
$$

Then

$$
\begin{aligned}
P(Z_{>} > \alpha = n^{1-\frac{1}{4}\epsilon_n^2/(k+1)}) &\leqslant \binom{\alpha}{\lambda}^{-1} E\left[\binom{Z_{>}}{\lambda}\right] \\
&\leqslant \binom{\alpha}{\lambda}^{-1}\binom{i}{\lambda} P\left(d_X(1) \geqslant (1+\epsilon_n)\frac{\log n}{k+1}\right)^{\lambda} \\
&\leqslant \left(\frac{i}{\alpha}e^{-\frac{1}{3}\epsilon_n^2\log n/(k+1)}(1+o(1))\right)^{\lambda} \\
&\leqslant [e^{-\frac{1}{12}\epsilon_n^2\log n/(k+1)}(1+o(1))]^{\lambda} \\
&= o(n^{-\log n}),
\end{aligned}
$$

where in the third inequality we made use of (22). The other half of (c) is proved in an identical way.

(d) We can be cruder with our estimates as we do not need a tight bound. Let $n_0 = n^{(2k+1)/(2k+3)}$. Then

$$
\begin{aligned}
P(|\mathrm{TINY}| \geqslant n_0) &\leqslant E\left[\binom{|\mathrm{TINY}|}{n_0}\right] \\
&\leqslant \binom{i}{n_0} P\left(d_X(1) \leqslant d = \frac{\alpha_k \log n}{k+1}+2\right)^{n_0}
\end{aligned}
$$

$$\leqslant \binom{i}{n_0}\left\{2\binom{j}{d}\left(\frac{1}{i}\right)^d\left(1-\frac{1}{i}\right)^{j-d}\right\}^{n_0}$$
$$\leqslant \left\{\frac{2ie}{n_0}\left(\frac{je}{di}\right)^d e^{-(j-d)/i}\right\}^{n_0}$$
$$\leqslant \{n^{2/(2k+3)}n^{\{1+o(1)\}/2(k+1)(k+2)}n^{-\{1-o(1)\}/(k+1)}\}^{n_0}$$
$$= o(n^{-\log n}). \quad \square$$

Armed with these results we can now prove Lemma 6.

Proof of Lemma 6 Fix i and j and consider $X = X_A(i,j)$. Condition on $\rho(X) \in [\nu|\boldsymbol{d}]^{\mu}$, where, by Lemmas 5 and 10 we have, with probability $1-o(n^{-\log n})$,

$$s \geqslant \nu \geqslant n - o(n^{k/(k+1)}), \tag{23a}$$

$$t \geqslant \mu \geqslant 2m - o(n^{k/(k+1)}), \tag{23b}$$

$$\boldsymbol{d} \in DX_0. \tag{23c}$$

Let us call DX_1 the set of $\boldsymbol{d}$ where (23a, b, c) are satisfied.

We work with the configuration model. We first consider the number of loops in $MG(X,k)$. Let $a = \lceil 10(\log n)^2 \rceil$ and $J \subset \text{EVENS} = \{2,4,6,\ldots,\mu\}$, where $|J| = a$, be given. Then (see (19a))

$$P(\psi(w_{i-1}) = \psi(w_i),\ i \in J) \leqslant \left(\frac{(\log n)^2}{\mu - 2a}\right)^a.$$

Hence

$$P(MG(X,k) \text{ has} \geqslant a \text{ loops}) \leqslant \binom{\frac{1}{2}\mu}{a}\left(\frac{(\log n)^2}{\mu-2a}\right)^a = o(n^{-\log n}). \tag{23d}$$

Next let $b = \lceil 7.5(\log n)^4 \rceil$. If $MG(X,k)$ contains at least b edges that are removed in the reduction to $G(X,k)$ then there exist disjoint sets $J_1, J_2 \subset$ EVENS with $|J_1| = |J_2| = b$ such that

$$\{\{\psi(w_{l-1}),\psi(w_l)\} : l \in J_2\} \subset \{\{\psi(w_{l-1}),\psi(w_l)\} : l \in J_1\}.$$

Thus

$$P(e_m(X) - e(X) \geqslant a+b) \leqslant o(n^{-\log n}) + \binom{\frac{1}{2}\mu}{b}^2\left\{2b\left(\frac{(\log n)^2}{\mu - 4b}\right)^2\right\}^b = o(n^{-\log n}). \tag{23e}$$

The result follows as there are $o(n^2)$ values for i and j. $\square$

The reader may have noticed that in order to apply Theorem 9 we need to show $\delta(G(X,k)) \geqslant k$ with high probability.

Lemma 11 (a) *Let i' and j' be as in Theorem* 9(a). *Then*

$$\Pi_1 = P[\delta(G(X,k)) < k \mid v(X) = n,\, e_m(X) = m] = o(1),$$

where $X = X(i', j')$.

(b) *Let i'' and j'' be as in Theorem* 9(b). *Then*

$$\Pi_2 = P[\delta(G(X,k)) < k \mid v(X) = n,\, e(X) = m] = o(1),$$

where $X = X(i'', j'')$.

Proof (a)

$$\Pi_1 = \sum_{\mathbf{d}\in DX(k,n,2m)} P[\delta(G(X,k)) < k \mid \rho(X) \in [n\,|\mathbf{d}]^{2m}] \times P[\rho(X) \in [n\,|\mathbf{d}]^{2m} \mid v(X) = n,\, e_m(X) = m].$$

Now (a) will follow from

$$P[\delta(G(X,k)) < k \mid \rho(X) \in [n\,|\mathbf{d}]^{2m}] = o(1) \qquad \text{if } \mathbf{d} \in DX_0, \quad (24)$$

since, using Theorem 9 and Lemma 10, we have

$$\sum_{\mathbf{d}\notin DX_0} P(\rho(X) \in [n\,|\mathbf{d}]^{2m} \mid v(X) = n,\, e_m(X) = m) \leqslant \frac{P(\rho(X) \in [n\,|\mathbf{d}]^{2m}, \text{ where } \mathbf{d} \notin DX_0)}{P(v(X) = n,\, e_m(X) = m)} = o(1).$$

We prove a set of results (25) that imply (24) and will be useful later. Assume that $\mathbf{d} \in DX_0$.

$P($there exists a loop or repeated edge within distance $10k$ of TINY in $MG(X,k) \mid \rho(X) \in [n\,|\mathbf{d}]^{2m}) = o(n^{-1/(k+2)})$. (25a)

$P($there exists a vertex incident with 3 edges of $MG(X,k)$ which are not in $G(X,k) \mid \rho(X) \in [n\,|\mathbf{d}]^{2m}) = O(n^{-2}\{\log n\}^{18})$. (25b)

Proof of (25a)

$$P(\text{there exists a loop}\ldots) \leqslant n^{(2k+1)/(2k+3)} \sum_{r=0}^{10k} n^r \left(\frac{(\log n)^4}{2m-2r-1}\right)^{r+1} = O(\{\log n\}^{30k+3} n^{-2/(2k+3)}).$$

P(there exists a repeated edge...)

$$\leq n^{(2k+1)/(2k+3)} \sum_{r=0}^{10k} n^{r+1}\left(\frac{(\log n)^4}{2m-2r-3}\right)^{r+2}$$
$$= O(\{\log n\}^{30k+6} n^{-2/(2k+3)}).$$

Proof of (25b)

$$P(\text{there exists a vertex...}) \leq P(\text{there exist } t+1 \text{ vertices inducing } t+3 \text{ edges in } MG(X,k) \text{ for some } 0 \leq t \leq 3)$$
$$\leq \sum_{t=0}^{3} n^{t+1}\left(\frac{(\log n)^4}{2m-2t-5}\right)^{t+3}$$
$$= O(n^{-2}\{\log n\}^{18}).$$

This completes the proof of (24) and thus of (a).

(b) For future reference, let E_2 denote the event specified by (25b). Let

$$DX_1(n) = \{(d_1, d_2, \ldots, d_n) \in DX_1\};$$
$$DX_2 = \left\{\boldsymbol{d} \in DX_1(n) : m \leq \sum_{i=1}^{n} d_i \leq m + 8(\log n)^4\right\}.$$

Define an equivalence relation $\approx$ on DX_2 by $\boldsymbol{d} \approx \boldsymbol{d}'$ if and only if $d_i' = d_{\sigma(i)}$ for some permutation σ of $[n]$. Let Ω be the set of equivalence classes of $\approx$. Then, letting $D_\omega = \bigcup_{d\in\omega} [n\,|\,\boldsymbol{d}]^{2m'}$ for $\omega \in \Omega$ and $2m' = \sum_{i=1}^{n} d_i$, we find that

$$\Pi_2 \leq \sum_{\omega\in\Omega} P[\delta(G(X,k)) < k \text{ and } \bar{E}_2 \mid \rho(X) \in D_\omega, e(X) = m]$$
$$\times P[\rho(X) \in D_\omega \mid v(X) = n,\ e(X) = m]$$
$$+ \frac{P\left(\rho(X) \in \bigcup_{d \notin DX_1} [n\,|\,\boldsymbol{d}]^{2m'}\right) + P(e_m(X) - e(X) > 8(\log n)^4) + P(E_2)}{P(v(X) = n,\ e(X) = m)}. \tag{26}$$

The calculations for (25b) are clearly valid for any i and j, so $P(E_2) = O(n^{-2}\{\log n\}^{18})$. Hence, by Theorem 9 and the proof of Lemma 6, the second term of (26) is $o(1)$.

Let us now fix $\omega \in \Omega$. We have the following inequalities:

$$P[\delta(G(X,k)) < k \text{ and } \bar{E}_2 \mid \rho(X) \in D_\omega,\, e(X) = m]$$
$$\leqslant nP(1 \in \text{TINY}, 1 \text{ is incident with a loop or multiple edge in } MG(X,k) \mid \rho(X) \in D_\omega,\, e(X) = m)$$
$$\leqslant n^{(2k+1)/(2k+3)} \max\{P(1 \text{ is incident with a loop or multiple edge in } MG(X,k) \mid \rho(X) \in [n\,|\mathbf{d}]^{2m'},\, e(X) = m) : \mathbf{d} \in \omega \text{ such that } 1 \in \text{TINY}\}.$$

If TINY $= \varnothing$ then $\delta(G(X,k)) < k$ and $\bar{E}_2$ is impossible, so we may assume that TINY $\neq \varnothing$. Fix $\mathbf{d} \in \omega$ for which $1 \in$ TINY.

Referring to the configuration model, let

$$\Phi_1 = \{F \in \Phi : 1 \text{ is incident with a loop or multiple edge, } \bar{E}_2 \text{ holds and } e(\hat{\varphi}(F)) = m\},$$

where $\hat{\varphi}(F)$ is the graph underlying the multigraph $\varphi(F)$, and

$$\Phi_2 = \{F \in \Phi : 1 \text{ is not incident with a loop or multiple edge, } \bar{E}_2 \text{ holds and } e(\hat{\varphi}(F)) = m\},$$

The result will follow from the following relation:

$$\frac{|\Phi_1|}{|\Phi_2|} = O\left(\frac{(\log n)^4}{n}\right). \tag{27}$$

To prove (27) we consider the bipartite graph $BG_1 = (\Phi_1, \Phi_2, \Lambda)$, where if $F_1 \in \Phi_1$ and $F_2 \in \Phi_2$ then $(F_1, F_2) \in \Lambda$ if there exists a vertex $v \neq 1$ such that

(i) v is not incident with any loops or multiple edges in $\varphi(F_1)$;
(ii) the distance from 1 to v in $\varphi(F_1)$ is at least 3;
(iii) $v \notin$ TINY;
(iv) F_2 is obtained from F_1 as follows: suppose that

$$W_1 = \{x_1, x_2, \ldots, x_p\} \quad \text{and} \quad W_v = \{y_1, y_2, \ldots, y_q\},$$

where $q \geqslant p$. If there is a loop (x_a, x_b) and $(y_1, w), (y_2, w') \in F_1$, then these pairs are replaced by $(x_a, w), (x_b, w'), (y_1, y_2) \in F_2$. If F_1 contains at most three pairs (x_a, z_a), (x_b, z_b), $\ldots$ $(a < b < \cdots)$, where $\{z_a, z_b, \ldots\} \subset W_t$ for some $t \neq 1$ and pairs (y_1, u_1), (y_2, u_2), $\ldots$, then we replace these pairs in F_2 by (x_a, u_1), (x_b, u_2), $\ldots$ and (y_1, z_a), (y_2, z_b), $\ldots$. All other pairs are in $F_1 \cap F_2$. (We interchange the loop/multiple edge pairings for W_1 and the beginning of W_v.)

It is straightforward to check that this is a proper definition. Furthermore (27) follows since $F_1 \in \Phi_1$ has degree at least

$$n - n^{(2k+1)/(2k+3)} - (\log n)^4 - 16(\log n)^4$$

in BG_1 and $F_2 \in \Phi_2$ has degree at most $16(\log n)^4$ in BG_1. □

3 Basic properties of random graphs of minimal degree at least *k*

We can now prove some properties of the random graph $G = G_{n,m}^{(k)} \in \mathcal{G}(n, m; \delta \geq k)$. Let

$$\text{SMALL} = \left\{ v \in [n] : d_G(v) \leq \frac{\alpha_k \log n}{k+1} \right\}; \qquad \text{LARGE} = [n] - \text{SMALL}.$$

Let

$$\text{PETIT} = \left\{ v \in [n] : d_G(v) \leq \frac{\log n}{3k+3} \right\};$$

$$N_G(S) = \{w \in V(G) - S : \exists\, v \in S \text{ with } \{v, w\} \in E(G)\}.$$

Lemma 12 *$G = G_{n,m}^{(k)}$ satisfies* (a) *to* (d) *below with probability $1 - o(n^{-A \log n})$ and* (e) *and* (f) *with probability $1 - o(1)$, where A is some positive constant.*

(a) $\Delta(G) \leq (\log n)^2$.

(b) $|\text{SMALL}| \leq 2n^{(2k+1)/(2k+3)}$ *and* $|\text{PETIT}| \leq n^{(4k+3)/(4k+4)}$.

(c) $S \subset \text{LARGE}$, $|S| \leq \dfrac{n}{2 \log n} \Rightarrow |N_G(S)| \geq \dfrac{\alpha_k \log n}{2k+2} |S|$.

(d) $S, T \subset [n]$, $|S| = |T| = n/\log\log n$ *and* $S \cap T = \emptyset$

$$\Rightarrow |\{(v, w) \in E(G) : v \in S,\ w \in T\}| \geq \frac{n \log n}{9(k+1)(\log\log n)^2}.$$

(e) *No connected subgraph of order at most $2k+5$ contains $k+2$ small vertices, i.e., vertices in* SMALL.

(f) *No small vertex is on a cycle of length at most $2k+2$.*

Proof Our proofs of (a)–(d) are on the following lines: let i'' and j'' be as in Theorem 9 (b) and let $X = X(i'', j'')$.

Now let Π and $\hat{\Pi}$ be properties such that

$$G(X, k) \in \Pi \Rightarrow \rho(X) \in \hat{\Pi} \text{ or } e_m(X) - e(X) > 8(\log n)^4. \qquad (28)$$

Then, on using Lemma 2, we have

$$
\begin{aligned}
&P(G_{n,m}^{(k)} \in \Pi) \\
&\quad = P[G(X,k) \in \Pi \mid v(X) = n,\, e(X) = m,\, \delta(G(X,k)) \geqslant k] \\
&\quad \leqslant P[\rho(X) \in \hat{\Pi} \mid v(X) = n,\, e(X) = m,\, \delta(G(X,k)) \geqslant k] \\
&\quad\quad + \frac{P(e_m(X) - e(X) > 8(\log n)^4)}{P[\delta(G(X,k)) \geqslant k \mid v(X) = n,\, e(X) = m] P[v(X) = n,\, e(X) = m]}. \qquad (29)
\end{aligned}
$$

Note that the second term on the right-hand side of (29) is $o(n^{-\frac{1}{2}\log n})$ by (23e) in the proof of Lemma 6, Lemma 11 and Theorem 9(b). Continuing,

$$
\begin{aligned}
P[\rho(X) &\in \hat{\Pi} \mid v(X) = n,\, e(X) = m,\, \delta(G(X,k)) \geqslant k] \\
&\leqslant \frac{P(\rho(X) \in \hat{\Pi} \mid v(X) = n,\, e(X) = m)}{P[\delta(G(X,k)) \geqslant k \mid v(X) = n,\, e(X) = m]} \\
&= (1 + o(1)) P(\rho(X) \in \hat{\Pi} \mid v(X) = n,\, e(X) = m) \qquad (30)
\end{aligned}
$$

by Lemma 11. Now

$$
\begin{aligned}
P(\rho(X) &\in \hat{\Pi} \mid v(X) = n,\, e(X) = m) \\
&\leqslant \sum_{d \in DX_1(n)} P(\rho(X) \in \hat{\Pi} \text{ and } \rho(X) \in [n|\boldsymbol{d}]^{\mu} \mid v(X) = n,\, e(X) = m) \\
&\quad + \frac{P(\rho(X) \notin \bigcup_{d \in DX_1} [\nu|\boldsymbol{d}]^{\mu})}{P(v(X) = n,\, e(X) = m)}. \qquad (31)
\end{aligned}
$$

Now, by the remark preceding (23) the second term on the right-hand side of (31) is $o(n^{-\frac{1}{2}\log n})$.

Furthermore,

$$
\begin{aligned}
P(\rho(X) &\in \hat{\Pi} \text{ and } \rho(X) \in [n|\boldsymbol{d}]^{\mu} \mid v(X) = n,\, e(X) = m) \\
&\leqslant \frac{P(\rho(X) \in \hat{\Pi} \mid \rho(X) \in [n|\boldsymbol{d}]^{\mu}) P(\rho(X) \in [n|\boldsymbol{d}]^{\mu})}{P(v(X) = n,\, e(X) = m)}. \qquad (32)
\end{aligned}
$$

Hence (29) to (32) imply

$$
\begin{aligned}
&P(G_{n,m}^{(k)} \in \Pi) \\
&\quad \leqslant \frac{(1 + o(1)) \max_{d \in DX_1(n)} \{P(\rho(X) \in \hat{\Pi} \mid \rho(X) \in [n|\boldsymbol{d}]^{\mu})\}}{P(v(X) = n,\, e(X) = m)} + o(n^{-\frac{1}{2}\log n}). \qquad (33)
\end{aligned}
$$

(a) Here we take $\hat{\Pi} = \{\rho(X) : \Delta(\rho(X) > (\log n)^2\}$ and use (33). It is easy to see that the first term is zero.

(b) For |SMALL| we take $\hat{\Pi} = \{\rho(X) : |\mathrm{TINY}| > n^{(2k+1)/(2k+3)}\}$ and again use (33). (Again the first term is zero.) For |PETIT| we would need a similar result for a set like TINY, but this is no problem.

(c) Here we take

$\hat{\Pi} = \{\rho(X) : \exists\, S, T \subset [n]$ such that

$$\text{(i)}\quad |S| \leqslant \frac{n}{2\log n},\ d_X(r) \geqslant \frac{\alpha_k \log n}{k+1} \text{ for } r \in S;$$

$$\text{(ii)}\quad T \subset [n]-S,\ |T| \leqslant \frac{\alpha_k \log n}{2k+2}|S|;$$

$$\text{(iii)}\quad N_{MG(X,k)}(S) \subset T\}.$$

Fix $\boldsymbol{d} \in DX_1(n)$ and let S and T satisfy (i) and (ii) above.

Let E^* be the event that each element of W_S is paired with an element of $W_{S\cup T}$ in the configuration model. Assume first that

$$|S| \leqslant s_0 = \frac{n}{4\alpha_k \mathrm{e}(k+1)(\log n)^3}.$$

Then

$$P(E^*) \leqslant \left\{\frac{1}{\mu-1}(\log n)^2\left(\frac{\alpha_k \log n}{2(k+1)}+1\right)|S|\right\}^{\{\alpha_k \log n/(k+1)\}|S|}$$
$$\leqslant \left(\frac{\alpha_k(\log n)^2}{n}|S|\right)^{\{\alpha_k \log n/(k+1)\}|S|}.$$

When $|S| > s_0$ we consider $\{t \in T : d_X(t) > (1+\epsilon_n)(\log n)/(k+1)\}$ and its complement in T. Then we find, letting $n_0 = n^{1-\epsilon_n^2/(4k+4)}$, that

$$P(E^*)$$
$$\leqslant \left\{\frac{1}{\mu-1}\left[(\log n)^2 n_0 + (1+\epsilon_n)\frac{\log n}{k+1}\left(\frac{\alpha_k \log n}{2(k+1)}+1\right)|S|\right]\right\}^{\{\alpha_k \log n/(k+1)\}|S|}$$
$$\leqslant \left(\frac{\alpha_k \log n}{(k+1)n}|S|\right)^{\{\alpha_k \log n/(k+1)\}|S|}.$$

Hence

$$P(\rho(X) \in \hat{\Pi} \mid \rho(X) \in [n|\boldsymbol{d}]^{\mu})$$
$$\leqslant \sum_{\sigma=1}^{s_0} \binom{n}{\sigma}\binom{n}{(\sigma\alpha_k \log n)/(2k+2)}\left(\frac{\alpha_k \sigma(\log n)^2}{n}\right)^{(\alpha_k \sigma \log n)/(k+1)}$$
$$+ \sum_{\sigma=s_0+1}^{n/2\log n} \binom{n}{\sigma}\binom{n}{(\sigma\alpha_k \log n)/(2k+2)}\left(\frac{\alpha_k \sigma \log n}{(k+1)n}\right)^{(\alpha_k \sigma \log n)/(k+1)}$$

$$\leqslant \sum_{\sigma=1}^{s_0} \left\{ \left(\frac{ne}{\sigma}\right)^{(k+1)/\alpha_k \log n} \left(\frac{2(k+1)en}{\alpha_k \sigma \log n}\right)^{1/2} \left(\frac{\alpha_k \sigma (\log n)^2}{n}\right) \right\}^{(\alpha_k \sigma \log n)/(k+1)}$$

$$+ \sum_{\sigma=s_0+1}^{n/2\log n} \left\{ \left(\frac{ne}{\sigma}\right)^{(k+1)/\alpha_k \log n} \left(\frac{2(k+1)en}{\alpha_k \sigma \log n}\right)^{1/2} \times \left(\frac{\alpha_k \sigma \log n}{(k+1)n}\right) \right\}^{(\alpha_k \sigma \log n)/(k+1)}$$

$$= o(n^{-(\alpha_k \log n)/(3k+3)}).$$

We can now apply (33).

(d) Here we take

$$\hat{\Pi} = \Big\{ \rho(X) : \exists\, S, T \subset [n] \text{ such that}$$

$$\text{(i) } |S| = |T| = \frac{n}{\log\log n} \text{ and } S \cap T = \varnothing;$$

$$\text{(ii) } MG(X,k) \text{ contains fewer than } \frac{n \log n}{8(k+1)(\log\log n)^2}\ S\text{–}T \text{ edges} \Big\}.$$

Fix $\boldsymbol{d} \in DX_1(n)$ and let S and T satisfy (i) of $\hat{\Pi}$. Consider the configuration model. Now

$$|W_S|, |W_T| \geqslant \frac{\{1-o(1)\} n \log n}{(k+1)\log\log n}.$$

Thus let $U \subset W_S$ be of size $(n \log n)/2(k+1)\log\log n$ and suppose that in the construction of F we first choose the pairs containing elements of U. Now, at any stage, if $x \in U$ then the probability it is paired with something in W_T is at least $(|W_T| - |U|)/\mu \geqslant 1/3\log\log n$. Hence the number of pairs stochastically dominates the binomial distribution $B(|U|, 1/3\log\log n)$. But

$$P\left[B\left(|U|, \frac{1}{3\log\log n}\right) \leqslant \frac{|U|}{4\log\log n}\right] \leqslant e^{-\frac{1}{32}|U|/3\log\log n}$$

$$\leqslant e^{-(n\log n)/200(k+1)(\log\log n)^2}.$$

Now

$$\frac{|U|}{4\log\log n} = \frac{n\log n}{8(k+1)(\log\log n)^2},$$

so

$$P(\rho(X) \in \hat{\Pi} \mid \rho(X) \in [n|\boldsymbol{d}]^{\mu}) \leq \binom{n}{n/\log\log n}^2 e^{-(n\log n)/200(k+1)(\log\log n)^2}$$
$$= o(n^{-\log n})$$

and thus (d) follows by (33).

(e) The proof of this (and (f)) is more complex than (a)–(d) as the failure probability in $G(X,k)$ is *not* $o(n^{-2})$ *in the unconditioned case.*

So let i' and j' be as in Theorem 9(a) and let $X = X(i',j')$. Let E_3 denote the event 'there is a vertex incident with 3 edges of $MG(X,k)$ which are not in $G(X,k)$, or there exists a loop or multiple edge incident with a vertex of degree k, $k+1$, $k+2$ or $k+3$ in $MG(X,k)$'.

Suppose that Π and $\hat{\Pi}$ are properties such that

$$MG(X,k) \in \hat{\Pi} \text{ and } \bar{E}_3 \Rightarrow G(X,k) \in \Pi. \tag{34}$$

Then

$$P[G(X,k) \in \Pi \mid v(X) = n,\ e_m(X) = m,\ \delta(G(X,k)) \geq k]$$
$$\geq P[MG(X,k) \in \hat{\Pi} \text{ and } \bar{E}_3 \mid v(X) = n,\ e_m(X) = m,\ \delta(G(X,k)) \geq k]$$
$$= P[MG(X,k) \in \hat{\Pi} \text{ and } \bar{E}_3 \mid v(X) = n,\ e_m(X) = m] - o(1)$$

(by applying Lemma 11)

$$= \sum_{\boldsymbol{d}\in DX_1(n)} P(MG(X,k) \in \hat{\Pi} \text{ and } \bar{E}_3 \mid \rho(X) \in [n|\boldsymbol{d}]^{2m})$$
$$\times P(\rho(X) \in [n|\boldsymbol{d}]^{2m} \mid v(X) = n, e_m(X) = m) - o(n^{2-\log n}) - o(1)$$

(by applying Lemma 10)

$$= \sum_{\boldsymbol{d}\in DX_1(n)} P(MG(X,k) \in \hat{\Pi} \mid \rho(X) \in [n|\boldsymbol{d}]^{2m})$$
$$\times P(\rho(X) \in [n|\boldsymbol{d}]^{2m} \mid v(X) = n,\ e_m(X) = m)$$
$$- o(n^{-1/(k+2)}) - O(n^{-2}\{\log n\}^{18}) - o(1)$$

(by applying (25a) and (25b))

$$\geq \min_{\boldsymbol{d}\in DX_1(n)} \{P(MG(X,k) \in \hat{\Pi} \mid \rho(X) \in [n|\boldsymbol{d}]^{2m})\} - o(n^{2-\log n}) - o(1). \tag{35}$$

On the other hand,

$$P(G(X,k) \in \Pi \mid v(X) = n,\ e_m(X) = m,\ \delta(G(X,k)) \geq k)$$
$$= P(G(X,k) \in \Pi \text{ and } e_m(X) - e(X) \leq 8(\log n)^4$$
$$\mid v(X) = n,\ e_m(X) = m,\ \delta(G(X,k)) \geq k) + o(1)$$

$$= \sum_{m'=m-8(\log n)^4}^{m} P(e(X) = m' \mid v(X) = n,\ e_m(X) = m,\ \delta(G(X,k)) \geqslant k) \times P(G^{(k)}_{n,m'} \in \Pi) + o(1) \quad (36)$$

on using Lemma 2. What we deduce from (35) and (36) is that if (34) holds then there exists an m' $(m-8(\log n)^4 \leqslant m' \leqslant m)$ such that

$$P(G^{(k)}_{n,m'} \in \Pi) \geqslant \min_{\mathbf{d} \in DX_1(n)} \{P(MG(X,k) \in \hat{\Pi} \mid \rho(X) \in [n \mid \mathbf{d}]^{2m})\} - o(1). \quad (37)$$

To prove (e) we let Π denote the property described in (e) and $\hat{\Pi}$ the equivalent property in $MG(X,k)$, where TINY takes the place of SMALL; it is easy to see that (34) holds. For a fixed $\mathbf{d} \in DX_1(n)$ we have

$$\begin{aligned} &P(MG(X,k) \notin \hat{\Pi} \mid \rho(X) \in [n \mid \mathbf{d}]^{2m}) \\ &\quad \leqslant \sum_{h=0}^{k+3} \binom{n^{(2k+1)/(2k+3)}}{k+2} \binom{n}{h} (k+2+h)^{h+k} \left(\frac{(\log n)^4}{2m-2(2k+5)} \right)^{k+h+1} \\ &\quad = o(1). \end{aligned}$$

Applying (37) we see that

$$P(G^{(k)}_{n,m'} \in \Pi) = 1 - o(1) \quad (38)$$

for some m' $(m-8(\log n)^4 \leqslant m' \leqslant m)$.

Now let $\tilde{\mathcal{G}}(n,m';\delta \geqslant k) = \mathcal{G}(n,m';\delta \geqslant k) \cap \Pi$. Set

$$g_{m'} = |\mathcal{G}(n,m';\delta \geqslant k)|; \qquad \tilde{g}_{m'} = |\tilde{\mathcal{G}}(n,m';\delta \geqslant k)|.$$

We can derive our result from (38) by showing

$$\frac{\tilde{g}_{m'+1}}{g_{m'+1}} \geqslant \frac{\tilde{g}_{m'}}{g_{m'}} \left\{ 1 - O\left(\frac{1}{n^{1/(k+2)}} \right) \right\} \quad (39)$$

for $m-8(\log n)^4 \leqslant m' \leqslant m$.

To prove (39) we consider the bipartite graph

$$H \equiv (\mathcal{G}(n,m';\delta \geqslant k),\ \mathcal{G}(n,m'+1;\delta \geqslant k),\ E(H)L),$$

where if

$$G_1 \in \mathcal{G}(n,m';\delta \geqslant k) \qquad \text{and} \qquad G_2 \in \mathcal{G}(n,m'+1;\delta \geqslant k)$$

then $(G_1,G_2) \in E(H)$ if and only if $E(G_1) \subset E(G_2)$. Also let $\tilde{H}$ denote the subgraph of H induced by $\tilde{\mathcal{G}}(n,m';\delta \geqslant k)$ and $\tilde{\mathcal{G}}(n,m'+1;\delta \geqslant k)$. Furthermore, let d and $\tilde{d}$ refer to degrees in H and $\tilde{H}$, respectively, so that, for example,

$$\sum_{G\in\mathcal{G}(n,m';\delta\geq k)} d(G) = \sum_{G\in\mathcal{G}(n,m'+1;\delta\geq k)} d(G). \tag{40}$$

Now if $G \in \mathcal{G}(n,m';\delta \geq k)$ then $d(G) = N-m'$ (where $N = \binom{n}{2}$) and if $G \in \mathcal{G}(n,m'+1;\delta \geq k)$ then $d(G) \geq m'+1-kn_k(G)$, where $n_k(G)$ is the number of vertices of degree k in G. Hence (40) implies

$$(N-m')g_{m'} \geq \{m'+1-kE_{m'+1}(n_k(G))\}g_{m'+1}, \tag{41}$$

where $E_{m'+1}(n_k(G))$ is the expectation over $\mathcal{G}(n,m'+1;\delta \geq k)$ of $n_k(G)$.

Applying the same ideas to $\tilde{H}$ we find that if $G \in \tilde{\mathcal{G}}(n,m';\delta \geq k)$ then $\tilde{d}(G) \geq N-m'-|\mathrm{SMALL}(G)|\Delta(G)^{10k}$ and if $G \in \tilde{\mathcal{G}}(n,m'+1;\delta \geq k)$ then $\tilde{d}(G) \leq m'+1$. We deduce then that

$$\{N-m'-E_{m'}(|\mathrm{SMALL}(G)|\Delta(G)^{10k})\}\tilde{g}_{m'} \leq (m'+1)\tilde{g}_{m'+1}. \tag{42}$$

Inequality (39) will follow from (41) and (42) once we prove

$$E_{m'}(|\mathrm{SMALL}(G)|\Delta(G)^{10k}) \leq 3n^{(2k+1)/(2k+3)}(\log n)^{20k} \tag{43}$$

for $m-8(\log n)^4 \leq m' \leq m$. But (43) follows since

$$P(|\mathrm{SMALL}(G^{(k)}_{n,m'})| \geq 2n^{(2k+1)/(2k+3)}) = o(n^{-A\log n}), \tag{44}$$

$$P(\Delta(G^{(k)}_{n,m'}) \geq (\log n)^2) = o(n^{-A\log n}), \tag{45}$$

where (44) and (45) are immediate from Lemma 12 (a) and (b).

(f) We proceed as in (e), taking Π to be the property described in (f) and obtain $\hat{\Pi}$ from Π by replacing SMALL by TINY as in (e). Then, for a fixed $\boldsymbol{d} \in DX_1(n)$,

$$\begin{aligned} &P(MG(X,k) \notin \hat{\Pi} \mid \rho(X) \in [n|\boldsymbol{d}]^{2m}) \\ &\quad\leq \sum_{h=3}^{2k+2} \binom{n}{h-1} n^{(2k+1)/(2k+3)} \frac{(h-1)!}{2}\left(\frac{(\log n)^4}{2m-2h}\right)^h \\ &\quad= o(1) \end{aligned}$$

The proof then continues in a manner analogous to (e). □

Let $\mathcal{G}_0 = \mathcal{G}_0(n,m;\delta \geq k)$ denote the members of $\mathcal{G}(n,m;\delta \geq k)$ having no k-spiders and satisfying conditions (e) and (f) of Lemma 12 and the following conditions (a)–(d), which are somewhat weaker than those of Lemma 12:

(a) $\Delta(G) \leq 2(\log n)^2$;

(b) $|\mathrm{SMALL}| \leq 3n^{(2k+1)/(2k+3)}$ *and* $|\mathrm{PETIT}| \leq 2n^{(4k+3)/(4k+4)}$;

(c) $S \subset \mathrm{LARGE}$; $|S| \leq \dfrac{n}{2\log n} \Rightarrow |N_G(S)| \geq \dfrac{\alpha_k \log n}{3k+3}|S|$;

(d) $S, T \subset [n]$, $|S| = |T| = n/\log\log n$ *and* $S \cap T = \emptyset$

$$\Rightarrow |\{(v,w) \in E(G) : v \in S,\ w \in T\}| \geqslant \frac{n \log n}{10(k+1)(\log\log n)^2}.$$

Let $\mathscr{G}_0' = \mathscr{G}_0'(n, m; \delta \geqslant k)$ be the set of graphs in $\mathscr{G}_0$ which satisfy the more stringent conditions of Lemma 12. From Lemma 12 and Lemma 13 below, it is easy to see that

$$\frac{|\mathscr{G}_0'|}{|\mathscr{G}_0|} = 1 - o(n^{-A\log n}). \tag{46}$$

4 k-spiders in random graphs

We now investigate the existence of k-spiders in $G_{n,m}^{(k)}$.

Lemma 13 *Let*

$$\theta_k = \frac{e^{-(k+1)c}}{(k+1)!\{(k-1)!\}^{k+1}(k+1)^{k(k+1)}}.$$

Then $\lim_{n\to\infty} P(G_{n,m}^{(k)}$ has a k-spider$) = 1 - e^{-\theta_k}$.

Proof We will not give all the details as (1) the case $k = 1$ is given in detail in [6] and (2) the most important ideas are already explained in Lemma 12 (e).

Call a k-spider *isolated* if it does not share any of its $k+2$ vertices with any other k-spider. By Lemma 12 (e) and (f) we have

$$\lim_{n\to\infty} P(G_{n,m}^{(k)} \text{ has a non-isolated } k\text{-spider}) = o(1).$$

We can therefore restrict our attention to isolated k-spiders. Let i' and j' be as in Theorem 9 (a) and let $X = X(i', j')$. The first task is to show that

$$\lim_{n\to\infty} P\big(MG(X,k) \text{ has an isolated } k\text{-spider} \mid \rho(X) \in [n\,|\boldsymbol{d}]^{2m}\big) = 1 - e^{-\theta_k} \tag{47}$$

for $\boldsymbol{d} \in DX_1(n)$. Thus let $\boldsymbol{d} \in DX_1(n)$ be fixed and let ζ denote the number of isolated k-spiders in $MG(X,k)$. If $p > 0$ is a fixed integer and $(\zeta)_p = \zeta(\zeta-1)\ldots(\zeta-p+1)$, we show that

$$\lim_{n\to\infty} E((\zeta)_p) = \theta_k^p \tag{48}$$

and then applying a basic result from probability theory (see, for example, [4], Theorem 1.20), we obtain (47).

Let

$$n_1 = \frac{1}{k!\,(k+1)^k} e^{-c} n^{k/(k+1)}.$$

It then follows, using (20), that

$$\begin{aligned} E((\zeta)_p) &\leq \binom{\{1+o(1)\}n_1}{k+1}^p \sum_{r=0}^{p} \binom{n}{r}\binom{o(n)}{p-r} p! \\ &\quad \times \left(\frac{k(1+\epsilon_n)\log n}{(k+1)\{2m-2(k+1)p\}}\right)^{r(k+1)} \left(\frac{k(\log n)^2}{2m-2(k+1)p}\right)^{(p-r)(k+1)} \\ &= \{1+o(1)\} \frac{n_1^{p(k+1)}}{(k+1)!^p} \binom{n}{p} p! \left(\frac{k\log n}{2(k+1)m}\right)^{(k+1)p} \\ &= \{1+o(1)\}\theta_k^p. \end{aligned} \tag{49}$$

A similar calculation shows that $E((\zeta)_p) \geq \{1-o(1)\}\theta_k^p$, so (48) follows.

We can now use (37) with Π and $\hat{\Pi}$ being the property of having an isolated k-spider to show that there exists m' such that

$$P(G_{n,m'}^{(k)} \text{ has an isolated } k\text{-spider}) \geq 1 - e^{-\theta_k} - o(1), \tag{50}$$

where $m - 8(\log n)^4 \leq m' \leq m$. Using a similar argument to the final argument of Lemma 12(e) we can replace m' by m in the probability inequality of (50).

To get the lower bound we again use (37) but this time with Π and $\hat{\Pi}$ being the property of *not* having a k-spider and then proceed as in Lemma 12(e). This yields

$$P(G_{n,m}^{(k)} \text{ has no isolated } k\text{-spider}) \geq e^{-\theta_k} - o(1)$$

and the result follows. □

Let S be a set of vertices of degree at least $k \geq 1$ is a graph G. For k even, an *S-system* is a set of $\frac{1}{2}k$ edge-disjoint systems of vertex-disjoint paths in G such that, for each system, every vertex in S is an internal vertex of some path of that system. For k odd, an *S-system* is a set of $\frac{1}{2}(k-1)$ edge-disjoint systems of paths, as before, together with an independent set of edges incident with each vertex in S, which is edge-disjoint from the path systems.

We shall need the following lemma.

Lemma 14 *Let G be a graph and let $S \subset V(G)$ consist of vertices of degree at least k. Denote by S_0 the set of vertices of S having degree k in G. Suppose that*

(a) *no vertex of G is adjacent to $k+1$ vertices in S_0;*
(b) *no connected subgraph of order at most $2k+3$ contains $k+2$ vertices of S;*
(c) *no vertex in S is on a cycle of length at most $2k+2$.*

Then G contains an S-system.

Proof Consider the subgraph induced by the edges incident with the vertices of S. Without loss of generality, we may take G to be this graph and, moreover, we can assume that G is connected. It is easy to show that $|S| \leqslant k+1$; otherwise we could construct a subtree of G with $k+2$ vertices of S and at most $2k+3$ vertices, contradicting (b). It follows that no cycle has length greater than $2k+2$ and thus, by (c), G must be a tree. Let T be the subtree obtained by deleting all leaves of G which are not in S. All leaves of T are in S, so $|V(T)| \leqslant 2k+1$. If $|S| = k+1$ and some vertex u is adjacent to every vertex in S, then some vertex w in S has degree at least $k+1$ in G, by (a). In this case consider $G-uw$; the component containing w clearly has an S-system, so we may assume that (a) holds with S_0 replaced by S. This, together with the observation that, for any vertex $v \in V(T)$, every component of $T-v$ contains a vertex of S, implies that $\Delta(T) \leqslant k$. (We are assuming that T is not trivial, otherwise the result is immediate.)

If $k = 1$ or 2 the result is immediate. If $k = 3$ then $|V(T)| \leqslant 7$ and it is easy to show that any maximal path in G containing a maximum number of vertices of S can be extended to an S-system.

Suppose then that $k \geqslant 4$ and that the assertion holds for smaller values of k. Let P_0 be a maximal path in T containing two vertices of highest degree in T. We claim that $\Delta(T-E(P_0)) \leqslant k-2$. Otherwise, T would contain three vertices of degree at least $k-1$ which were not on a path. But then T would have at least

$$3(k-2) > k+1 \geqslant |S|$$

leaves, which is impossible.

Now extend P_0 to a maximal path P in G. Every vertex w of S not on P is adjacent to at least two leaves of G; this gives a path P_w of length two through w. We will show that the path system $P^* = \{P\} \cup \{P_w : w \in S-V(P)\}$ can be extended to an S-system. Let $G_0 = G-E(P^*)$. Then G_0 is a forest, two components of which contain just one vertex of S, and no vertex not in S is of degree greater than $\Delta(T-E(P_0)) \leqslant k-2$. Thus G_0 trivially satisfies the conditions of the lemma with k replaced by $k-2$. So, by our assumption, G_0 contains an S-system, which, together with P^*, yields an S-system for G. □

5 The colouring argument

It follows from Lemma 14 that if $G \in \mathscr{G}_0$ then SMALL is *k-coverable* in the following sense: there exist sets of vertex-disjoint paths $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_{\lfloor k/2 \rfloor}$ plus a matching $\mathscr{P}_{\lceil k/2 \rceil}$ if k is odd, where

(i) both endpoints of every path in $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_{\lfloor k/2 \rfloor}$ are in LARGE;

(ii) each element of SMALL is an internal vertex of one path in each of $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_{\lfloor k/2 \rfloor}$;

(iii) if k is odd then each element of SMALL is incident with an edge of $\mathscr{P}_{\lceil k/2 \rceil}$;

(iv) (minimality) if EP_i denotes the edges of $\mathscr{P}_i$ ($i = 1, 2, \ldots, \lceil \frac{1}{2}k \rceil$) then $e \in \bigcup_{i=1}^{\lceil k/2 \rceil} EP_i$ implies $e \cap \text{SMALL} \neq \emptyset$;

(v) the edge sets $EP_1, EP_2, \ldots, E_{\lceil k/2 \rceil}$ are pairwise disjoint.

For each $G \in \mathscr{G}_0$ choose fixed sets of paths $\mathscr{P}_1, \mathscr{P}_2, \ldots,$ together with a matching if k is odd. Let $EP_i(G)$ refer to this fixed choice.

Suppose $H_1, H_2, \ldots, H_r$ is a sequence of edge-disjoint Hamilton cycles of G. They are said to be *compatible* if $H_i \supset EP_i$ for $i = 1, 2, \ldots, r$, where $r \leqslant \lfloor \frac{1}{2}k \rfloor$.

We are now close to proving our main theorem. The main tools will be Pósa's theorem [13] plus the colouring argument of Fenner and Frieze [9]. To use the colouring argument we need to consider the deletion of a small set of edges. So if $G = G_{n,m}^{(k)}$ and $X \subseteq E(G)$ we say that X is *deletable* if

(i) X is a matching;

(ii) X is not incident with any vertex of PETIT.

Let G_X be the graph $G - X$ obtained by deleting X from G.

We shall need to consider the state of G after deleting some edge-disjoint Hamilton cycles. Thus we prove the following result.

Lemma 15 *Let $G \in \mathscr{G}_0$; let X be deletable and $H_1, H_2, \ldots, H_r$, $r < \lceil \frac{1}{2}k \rceil$, be compatible edge-disjoint Hamilton cycles. Let*

$$K = G - \left(X \cup \bigcup_{i=1}^{r} E(H_i) \right)$$

be the graph obtained by deleting X and the edges of these cycles, and let $\Sigma = \text{SMALL} \cup N_G(\text{SMALL})$.

(a) *If $\emptyset \neq S \subseteq$ LARGE and $|S| \leqslant \beta_k n$ then*

$$|N_K(S) - \Sigma| \geqslant 3|S|,$$

where $\beta_k = \alpha_k / 7(k+1)(2k+5)$.

(b) *If $r < \lfloor \frac{1}{2}k \rfloor$ then K is connected.*

Proof (a) By Lemma 12 (e) and (f) we have

$$|N_G(S) \cap \Sigma| \leqslant (k+1)|S|. \tag{51}$$

Let $S_1 \subseteq S$ be of size $\min\{|S|, \lfloor n/2\log n \rfloor\}$. Then

$$\begin{aligned} |N_K(S)| &\geqslant |N_G(S)| - (2r+1)|S| \\ &\geqslant |N_G(S_1)| - (2r+2)|S| \\ &\geqslant \frac{\alpha_k \log n}{3(k+1)}|S_1| - (2r+2)|S| \\ &\geqslant (2k+5)|S| - (2r+2)|S| \\ &\geqslant (k+4)|S|. \end{aligned} \tag{52}$$

The result now follows after using (51).

(b) Suppose S is a component of K ($|S| \leqslant \frac{1}{2}n$) and suppose first that $|S| \leqslant \beta_k n$. Since $E(K) \supseteq EP_{\lfloor k/2 \rfloor}$ we have $S_1 = S \cap \text{LARGE} \neq \emptyset$. Thus

$$\begin{aligned} |N_K(S)| &\geqslant |N_K(S_1)| - |N_K(S_1) \cap \text{SMALL}| \\ &\geqslant (k+4)|S_1| - (k+1)|S_1| > 0, \end{aligned}$$

where we have used (51) and (52) to establish the second inequality. Thus $|S| \leqslant \beta_k n$ is not possible, and condition (d) of the definition of $\mathcal{G}_0$ shows that we have not deleted enough edges to create a component of size greater than $\beta_k n$. □

We now proceed as in the proof of Theorem 1.2 of [6].

Let $\mathcal{A}_k$ be the graph property of having $\lfloor \frac{1}{2}k \rfloor$ edge-disjoint Hamilton cycles plus a further edge-disjoint matching of size $\lfloor \frac{1}{2}n \rfloor$ if k is odd. Let r be a non-negative integer and $G \in \mathcal{G}_0$. Let $H_1, H_2, \ldots, H_r$ be a compatible sequence of edge-disjoint Hamilton cycles. $G - \bigcup_{i=1}^r E(H_i)$ is called an *r-subgraph* of G.

Let now $\phi(G) = (r, s)$, where r is the maximal length of a compatible sequence of edge-disjoint Hamilton cycles and

$$s = \begin{cases} 0 & \text{if } k = 2r, \\ \text{maximal cardinality of a matching containing} & \\ EP_{\lceil k/2 \rceil} \text{ in any } r\text{-subgraph of } G & \text{if } k = 2r+1, \\ \text{maximal length of a path } P \text{ in any } r\text{-subgraph} & \\ \text{of } G \text{ such that } Q \in \mathcal{P}_{r+1} \text{ implies } Q \text{ is a} & \\ \text{subpath of } P \text{ or } Q \text{ is disjoint from } P & \text{if } k \geqslant 2r+2. \end{cases}$$

Thus if $\phi(G) = \theta(k,n) = (\lfloor \frac{1}{2}k \rfloor, \lfloor \frac{1}{2}n \rfloor (k - 2\lfloor \frac{1}{2}k \rfloor))$ then $G \in \mathcal{A}_k$.

If $\phi(G) = (r,s)$ we define a *ϕ-subgraph* of G to be any r-subgraph containing either a matching of size s or a path of length s as the case may be.

Lemma 16 *Suppose $G \in \mathscr{G}_0 - \mathscr{A}_k$ and X is deletable. Let $u = \lceil \beta_k n \rceil$ and $\phi(G) = (r,s)$. Then for n large there exist a ϕ-subgraph H of G_X, $A = \{a_1, a_2, \ldots, a_t\}$, $A_1, A_2, \ldots, A_t \subseteq [n]$ $(t > u)$ such that, for $i = 1,2,\ldots,t$, $|A_i| \geqslant u$, $a_i \notin A_i$ and, if $a \in A_i$, then $e = \{a, a_i\} \notin E(H)$ and $\phi(H+e) \neq \phi(H)$.*

Proof Let, in fact, H be any ϕ-subgraph of G_X. Suppose first that $k = 2r+1$. Let

$$\mathscr{M} = \{M : M \supseteq EP_{\lceil k/2 \rceil} \text{ and } M \text{ is a matching of size } s \text{ in } H\};$$

$$A = \{a : a \text{ is left exposed by some matching in } \mathscr{M}\} = \{a_1, a_2, \ldots, a_t\};$$

$$A_i = \{a : a \text{ and } a_i \text{ are left exposed by some matching in } \mathscr{M}\} \subseteq A \qquad (i = 1,2,\ldots,t).$$

Clearly $a \in A_i$ implies $e = \{a, a_i\} \notin E(G_X)$ and $\phi(G_X+e) \neq \phi(G_X)$. We must show that $t \geqslant u$. Consider, for example, A_1. Let $A_1' = A_1 \cap \text{LARGE}$. $A_1' \neq \emptyset$, else a_1 is the only vertex left exposed by any matching in $\mathscr{M}$, contradicting $G \notin \mathscr{A}_k$. We will show that

$$|N_H(A_1') - \Sigma| < |A_1'| \tag{53}$$

and then Lemma 15 (a) implies that $|A_1'| \geqslant u$.

To prove (53), let $\{x,y\} \in E(H)$, $x \in A_1'$ and $y \notin \Sigma$, and let $M \in \mathscr{M}$ leave x and a_1 exposed. y is not exposed, so let $\{y,z\} \in M$. Then $z \in \text{LARGE}$ (otherwise $y \in \Sigma$) and, because $M' = M + \{x,y\} - \{y,z\}$ leaves a_1 and z exposed, we have that $z \in A_1'$. Thus $y \in N_H(A_1') - \Sigma$ implies that y is adjacent to A_1' via an edge of M and (53) follows.

Assume next that $k > 2r+1$. If $P = (v_1, v_2, \ldots, v_k)$ is a path of H and $e \equiv \{v_k, v_i\} \in E(H)$ then, as has often been done before, we consider the path

$$\text{ROTATE}(P,e) = (v_1, v_2, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}) \tag{54}$$

and let v_1 be called the *fixed endpoint* in this construction.

Now let P be a path of length s in H either wholly containing or wholly disjoint from any path of $\mathscr{P}_{r+1}$. Let a_1 be one endpoint of P. Consider all paths that may be obtained from P by a sequence of rotations with a_1 as a fixed endpoint, subject to the restriction that no rotation is allowed in which the deleted edge ($\{v_i, v_{i+1}\}$ in (54)) is in EP_{r+1}.

Let A_1 be the set of endpoints, other than a_1, of the paths produced by this procedure and let $A_1' = A_1 \cap \text{LARGE}$. $A_1' \neq \varnothing$ as vertices in SMALL can only be internal vertices of any of the paths produced. Following Pósa [13] we show that

$$|N_H(A_1') - \Sigma| < 2|A_1'| \tag{55}$$

and then Lemma 15 (a) shows that $|A_1'| \geqslant u$. Note also that there is no edge $\{a, a_1\} \in E(H)$ where $a \in A_1$, since, by Lemma 15 (b), H is connected and such an edge could be used to give a longer path.

So let $\{x, y\} \in E(H)$, $x \in A_1'$ and $y \notin \Sigma$ (y is an internal vertex of P as P is maximal). We observe as in [6] that y has at least one neighbour in A_1' on P and (55) follows.

Finally, take $A = A_1 \cup \{a_1\}$ and repeat the argument for $a \in A_1$ using any path of length s with a as fixed endpoint. □

We now give a final lemma which essentially proves our main theorem.

Lemma 17

$$\lim_{n\to\infty} P(G_{n,m}^{(k)} \in \mathscr{A}_k \mid G_{n,m}^{(k)} \text{ has no } k\text{-spider}) = 1.$$

Proof Let

$$\mathscr{G}_2 = \mathscr{G}_2(n, m; \delta \geqslant k) = \mathscr{G}_0 - \mathscr{A}_k.$$

If $G \in \mathscr{G}(n, m; \delta \geqslant k)$ and $X \subseteq E(G)$, let X be *strongly deletable* if X is deletable and $\phi(G_X) = \phi(G)$. Let $\omega = \lceil\sqrt{\log n}\rceil$ and for $X \subseteq E(G)$ ($|X| = \omega$) let

$$a(X, G) = \begin{cases} 1 & \text{if } G \in \mathscr{G}_2 \text{ and } X \text{ is strongly deletable,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $G \in \mathscr{G}_2$. We show first that

$$\sum_{\substack{X \subset E(G) \\ |X| = \omega}} a(X, G) > \binom{m}{\omega}\left(1 - \frac{2k^2}{\log n}\right)^{\omega}.$$

To see this let $\phi(G) = (r, s)$. Choose a random ω-subset X of $E(G)$. It is easy to see that the right-hand side of (55) is a lower bound for the expected number of X satisfying (1) X is a matching (use Lemma 12 (a)), (2) X is not incident with a vertex of PETIT (use Lemma 12 (b)) and (3) X contains no edge of some fixed compatible sequence of r Hamilton cycles and s further edges.

If (1), (2) and (3) hold then $a(X,G) = 1$. Hence

$$\sum_{G\in\mathscr{G}_2}\sum_{\substack{X\subseteq E(G)\\|X|=\omega}} a(X,G) \geqslant \binom{m}{\omega}\left(1-\frac{2k^2}{\log n}\right)^{\omega}|\mathscr{G}_2|. \tag{56}$$

We now bound the sum on the left-hand side of (56) from above. Let

$$\Omega = \{H : \exists G \in \mathscr{G}_0,\ X \subseteq E(G),\ |X|=\omega,\ X \text{ is not incident with any vertex of PETIT}(G), \text{ and } H = G_X\}.$$

For $H \in \Omega$ let $S_H = |\{X : G = H+X \in \mathscr{G}_2 \text{ and } a(X,G) = 1\}|$. We show

$$S_H \leqslant \binom{\binom{n}{2}-\gamma n^2}{\omega}, \tag{57}$$

where $\gamma = \frac{1}{2}\beta_k^2$ and β_k is as in Lemma 15. To see this note that if $S_H > 0$ then H contains sets $A, A_1, A_2, \dots, A_t$ as in Lemma 16 and $X \in S_H$ implies (through $a(X,G) = 1$) that X contains no edge of the form $\{a, a_i\}$, where $a_i \in A$ and $a \in A_i$ $(i = 1,2,\dots,t)$. This implies (57).

But

$$\sum_{G\in\mathscr{G}_2}\sum_{\substack{X\subset E(G)\\|X|=\omega}} a(X,G) = \sum_{H\in\Omega} S_H.$$

Thus (56) and (57) imply

$$|\mathscr{G}_2| \leqslant \binom{\binom{n}{2}-\gamma n^2}{\omega}\binom{m}{\omega}^{-1}\left(1-\frac{2k^2}{\log n}\right)^{-\omega}|\Omega|. \tag{58}$$

Clearly we must produce an upper bound for $|\Omega|$. To do this we count pairs (H,X), where (1) H is a graph with $V(H) = [n]$, (2) $|E(H)| = m-\omega$, (3) $X \subset [n]^{(2)} - E(H)$ and $|X| = \omega$, (4) $G = H+X \in \mathscr{G}_0$ and (5) X is not incident with any vertex in PETIT(G); in this case we call (H,X) a *proper* pair. Let ζ be the number of such pairs. Clearly

$$\zeta = \{1-o(1)\}\binom{m}{\omega}|\mathscr{G}_0| \tag{59}$$

as $(G-X,X)$ is almost always a proper pair when $G \in \mathscr{G}_0$, $|X| = \omega$ and $X \subseteq E(G)$. On the other hand

$$\zeta = \sum_{H\in\Omega} \epsilon_H, \tag{60}$$

where $\epsilon_H = |\{X : X \text{ is a proper partner for } H\}|$ for $H \in \Omega$. Let

$$\Omega' = \{H \in \Omega : \exists \text{ a proper partner } X \text{ for which } H+X \in \mathcal{G}_0'\}|.$$

We observe that

$$H \in \Omega' \Rightarrow \epsilon_H \geqslant \{1-o(1)\}\binom{\binom{n}{2}-m+\omega}{\omega}. \tag{61}$$

To see this let $H \in \Omega'$ and (H,X) be proper with $H+X \in \mathcal{G}_0'$. Let $Y \subseteq [n]^{(2)} - E(H)$ and suppose that the edges in Y are not incident with any vertex within $10k$ of $\mathrm{PETIT}(H+X)$. Observe that

$$\mathrm{SMALL}(H+X) = \mathrm{SMALL}(H+Y).$$

It is easy then to see that $H+Y \in \mathcal{G}_0$ (but not necessarily $\mathcal{G}_0'$). A simple calculation yields (61). It follows from (59) to (61) that

$$|\mathcal{G}_0| \geqslant \{1-o(1)\}|\Omega'|\binom{\binom{n}{2}-m+\omega}{\omega}\Big/\binom{m}{\omega}. \tag{62}$$

We will show that

$$|\Omega'| = \{1-o(1)\}|\Omega|, \tag{63}$$

to obtain from (58) and (62) that

$$\frac{|\mathcal{G}_2|}{|\mathcal{G}_0|} \leqslant \{1+o(1)\}\binom{N-\gamma n^2}{\omega}\Big/\binom{N-m+\omega}{\omega} = o(1).$$

which proves the lemma.

If $H \in \Omega - \Omega'$ then there exist $G \in \mathcal{G}_0 - \mathcal{G}_0'$ and $X \subseteq E(G)$ ($|X| = \omega$) such that $H = G - X$. Hence

$$|\Omega| - |\Omega'| \leqslant \binom{m}{\omega}(|\mathcal{G}_0| - |\mathcal{G}_0'|). \tag{64}$$

On the other hand,

$$|\Omega'| \geqslant |\mathcal{G}_0'|\Big/\binom{N-m+\omega}{\omega} \tag{65}$$

as, given $G \in \mathcal{G}_0'$, one can always find a proper pair (H,X) such that $H+X = G$.

From (64) and (65) we obtain

$$\frac{|\Omega|}{|\Omega'|} - 1 \leqslant \binom{m}{\omega}\binom{N-m+\omega}{\omega}\left(\frac{|\mathcal{G}_0|}{|\mathcal{G}_0'|} - 1\right)$$
$$= o(n^{4\omega} n^{-A\log n}) = o(1),$$

which is equivalent to (63), with the last relation following from (46). □

6 The main theorem

We now come finally to the main result of our paper.

Theorem 18 *Let*

$$m = \tfrac{1}{2}n\left(\frac{\log n}{k+1} + k\log\log n + c_n\right).$$

Then

$$\lim_{n\to\infty} P(G_{n,m}^{(k)} \in \mathcal{A}_k) = \begin{cases} 0 & \text{if } c_n \to -\infty \text{ sufficiently slowly,} \\ e^{-\theta_k} & \text{if } c_n \to c, \\ 1 & \text{if } c_n \to +\infty. \end{cases}$$

Proof When $c_n \to c$ we need only consult Lemmas 13 and 17. For $c_n \to \pm\infty$ we have to rework all the calculations. For $c_n \to +\infty$ things get easier, but we can only allow $c_n \to -\infty$ so that

$$\frac{1}{k+1} + \frac{c_n}{\log n} \geqslant \epsilon$$

for some fixed $\epsilon > 0$. In this case our methods work. □

The above theorem describes rather neatly the secondary obstruction to membership of $\mathcal{A}_k$. In a related vein one could consider $P(G_{n,m}^{(k+1)} \in \mathcal{A}_k)$. We strongly believe and seem close to proving that there exist constants $c_1, c_2, \ldots$ such that if $c \geqslant c_k$ then

$$\lim_{n\to\infty} P(G_{n,cn}^{(k+1)} \in \mathcal{A}_k) = 1.$$

Finally we mention that we could have based our proof on an analysis of HAM in [5]. This would show that the Hamilton cycles and matchings are almost always constructable in polynomial time. The proof, however, would have been even longer.

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