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Random Graph Orders

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Abstract. Let P_n be the order determined by taking a random graph G on $\{1, 2, \ldots, n\}$, directing the edges from the lesser vertex to the greater (as integers), and then taking the transitive closure of this relation. We call such an ordered set a *random graph order*. We show that there exist constants c, and σ, such that the expected height and set up number of P_n are sharply concentrated around cn and σn respectively. We obtain the estimates: $.565 < c < .610$, and $.034 < \sigma < .289$. We also discuss the width, dimension, and first-order properties of P_n.

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1. Introduction

The application of probabilistic methods in graph theory, and the study of random graphs, introduced almost 30 years ago by Erdös and Renyi, now seem commonplace. Random orders however have received less attention, in part because the presence of transitivity prevents the completely independent choice of related pairs which is so effective in random graphs. On the other hand, this very difficulty means that a number of alternative methods for generating 'random' partial orders exist, one of which we propose to study here. In [11] and [12], Peter Winkler studies a method, for generating random orders of bounded dimension by selecting points uniformly and independently from the k-dimensional unit cube with the usual coordinatewise order. In his paper, a number of results concerning such orders, for instance the expected number of minimal elements, expected height and width, and some first-order properties are investigated. Here we present a similar study of a different method for generating random partial orders.

It should be noted that if one simply enumerates the partial orders of size n uniformly then one obtains the result 'almost every order is of height three'. This is one example where mathematical reality clashes strongly with ones natural feeling for the subject. Asked to imagine a 'random' partial order on a set of say 100 elements that the height should equal three seems rather less than likely. Nonetheless, there has been some very interesting work done with respect to this measure, including a recent proof of a first order 0-1 law for finite orders due to Compton [3].

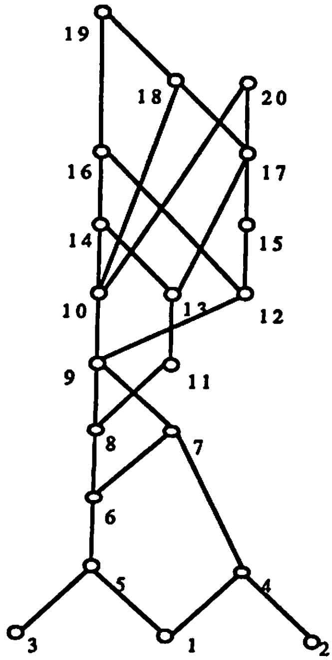

Fig. 1. Random graph order of size 20.

Recall the basic definition of a random graph (with $p = \frac{1}{2}$) of size n, namely a graph on the vertices $\{1, 2, \ldots, n\}$ where the existence of an edge between i and j for each unordered pair $\{i, j\}$ is determined independently with probability $\frac{1}{2}$. Our method of constructing a random ordered set of size n is to simply form such a random graph, then view each edge ij with $i < j$ as a directed edge from i to j, and then to take the transitive closure of the resulting relation as our (strict) order relation. Another way to construct such an order is to view this process sequentially. At step k we add an element k, choosing independently which already existing elements k will lie above, and then letting transitivity do the rest. Of course, some of our choices will be redundant, but we do not concern ourselves with that. Thus, the labelled Hasse diagram of such an ordered set can be drawn 'from the bottom up'. In Figure 1, we show a 20 element ordered set constructed by this method. We refer to the orders constructed this way as *random graph orders*. We will obtain estimates on the width, height, set up number, and dimension of random graph orders of size n.

2. Height

To foreshadow briefly the main result of this section is to show that random graph orders are tall. Specifically, the average height of a random graph order of size n is roughly $.6n$.

We wish to find the height of the average random graph order of size n (in the limit as n tends to infinity). To this end, we think of a random graph order being constructed sequentially as indicated above, and define two sequences of random variables giving the length of the longest chain and the number of endpoints of longest chains at each stage. More precisely

l_k = length of the longest chain in P_k,
$\#_k$ = number of points in P_k which are endpoints of a longest chain.

We will obtain fairly sharp estimates for the expected value of l_n by defining other random variables which bound l_k, and $\#_k$ from either side.

To obtain a lower bound for l_k consider the following pessimistic (in terms of constructing a longest chain) procedure for adding a $(k+1)$st element to a random graph order. We know that the existing longest chain(s) will be extended by one link, and the value of $\#$ reduced to 1 with probability: $1-(\frac{1}{2})^{\#_k}$. However, we assume pessimistically that if this does not occur then $\#$ will be incremented by 1 with probability only $\frac{1}{2}$ (this amounts to assuming that all the longest chains have a common penultimate vertex). This gives the following recurrence for random variables λ and σ which are underestimates for l and $\#$, respectively:

$$(\lambda_{k+1}, \sigma_{k+1}) = \begin{cases} (\lambda_k + 1, 1) & \text{with probability } 1-(\frac{1}{2})^{\sigma_k}, \\ (\lambda_k, \sigma_k + 1), & \text{with probability } (\frac{1}{2})^{\sigma_k+1}, \\ (\lambda_k, \sigma_k), & \text{with probability } (\frac{1}{2})^{\sigma_k+1}. \end{cases}$$

If we consider only the second coordinate of the above recurrence, then we obtain a recurrent Markov process (see, e.g., [8]) and so $p_j = \lim_{n\to\infty} \Pr(\sigma_n = j)$ exists. Furthermore, we obtain the following recurrence

$$p_{j+1} = \frac{1}{2^{j+1}} p_j + \frac{1}{2^{j+2}} p_{j+1} (j \geqslant 1) \quad \text{or} \quad p_{j+1} = \frac{2}{2^{j+2}-1} p_j \; (j \geqslant 1).$$

In turn, this yields

$$p_{j+1} = \left(\prod_{i=1}^{j} \frac{2}{2^{i+2}-1}\right) p_1.$$

Then, using the fact that the sum of the p's must be 1, we can produce the values shown in Table I.

Now the expected (long run) increment in the path length is simply

$$\sum_{j=1}^{\infty} \left(1 - \frac{1}{2^j}\right) p_j = .5654.$$

Since everything began as an underestimate, we may safely say that for sufficiently large n, the expected length of the longest chain in P_n is at least $.5654n$.

Now we wish to adopt an optimistic viewpoint. The initial part of the analysis is as before only now we assume that if we fail to increment the chain length by one, then we always increment the number of endpoints of the longest chains

Table I. Values of p_j for small j

j	p_j
1	.7540
2	.2154
3	2.872×10^{-2}
4	1.853×10^{-3}
5	5.883×10^{-5}
6	9.264×10^{-7}
7	7.266×10^{-9}
8	2.844×10^{-11}
9	5.560×10^{-14}
10	5.432×10^{-17}
$\vdots$	
15	$< 10^{-35}$

by one. This gives the modified recurrence:

$$(\lambda_{k+1}, \sigma_{k+1}) = \begin{cases} (\lambda_k + 1, 1) \text{ with probability } 1 - (\frac{1}{2})^{\sigma_k}, \\ (\lambda_k, \sigma_k + 1) \text{ with probability } (\frac{1}{2})^{\sigma_k}. \end{cases}$$

Of course, this simplifies the recurrence for the long run probabilities somewhat and we get:

$$p_{j+1} = \frac{1}{2^j} p_j \quad (j \geqslant 1),$$

that is

$$p_{j+1} = 2^{-j(j+1)/2} p_1.$$

An evaluation of the (overestimate) of the expected length of the longest chain in the case gives $.610n$. It is very easy to modify these recurrences to allow an edge probability other than $\frac{1}{2}$ (but still constant). Doing so yields the table of upper and lower bounds for the expected length of the longest path in P_n over n (Table II).

Table II. Table of upper and lower bounds

Edge probability	Lower bound	Upper bound
.01	.0138	.0799
.1	.133	.256
.2	.255	.367
.3	.366	.456
.4	.469	.535
.5	.565	.610
.6	.654	.682
.7	.739	.753
.8	.822	.828
.9	.907	.909
.99	.990097	.990098

The above analysis shows that if H_n is the height of a random random graph order ($p = 1/2$) then:

$$\lim_{n\to\infty} \Pr(.5654n < H_n < .610n) = 1.$$

We continue by showing that there is some constant c, such that H_n is sharply concentrated with $E(H_n) = cn$. The main tool in the proof is the use of a martingale inequality due to Azuma [1] and Hoeffding [7], but see especially in a context similar to ours, Shamir and Spencer [10]. We express it in the following form.

LEMMA 1. *Let $X_1, X_2, \dots, X_n$ be random objects and let $M = M(X_1, X_2, \dots, X_n)$. Let $X^{(i)} = (X_1, X_2, \dots, X_i)$, and let $M_i = M_i(X^{(i)}) = E(M \mid M^{(i)})$ so that $M_0 = E(M)$ and $M_n = M$. Let*

$$\delta_i = \sup_{X^{(i)}} (|M_i(X^{(i)}) - M_{i-1}(X^{i-1)})|), \quad i = 1, 2, \dots, n.$$

Then

$$\Pr(|M - E(M)| \geq t) \leq 2\exp\left(-\frac{t^2}{\sum_{i=1}^{n} \delta_i^2}\right).$$

Using this lemma we shall prove

THEOREM 2. *There exists a constant c, $.5654 < c < .610$ such that*:

(a) *if $\varepsilon > 0$ is fixed then*

$$\Pr(|H_n - cn| \geq \varepsilon n) = O(\exp(-\varepsilon^2 n/72(\log n)^2),$$

(b) $\Pr\left(\lim_{n\to\infty} \frac{H_n}{n} = c\right) = 1.$

Proof. (b) follows from (a) and the Borel–Cantelli Lemma (see [5]) since:

$$\sum_{n=1}^{\infty} \Pr(|H_n - cn| \geq \varepsilon n) < \infty, \tag{1}$$

and so we concentrate on proving (a).

It is temporarily convenient to introduce the random variable L_n equal to the longest path from 1 to n (referred to as a $1-n$ chain). Let now H'_{n-1} be the length of the longest chain contained in $\{2, 3, \dots, n-1\}$ and observe that

$$H'_{n-2} \text{ is distributed as } H_{n-2}. \tag{2}$$

We can easily prove that

$$L_n \leq H'_{n-2} + 2, \tag{3a}$$

$$\Pr(H'_{n-2} - L_n \geq 2s) \leq (\tfrac{1}{2})^{s-1}. \tag{3b}$$

Formula (3a) is trivial, and to prove (3b) assume that the longest chain in $\{2, 3, \dots, n-1\}$ is $i_1 < i_2 < \cdots < i_k$, where $k \geq 2s$. If $H'_{n-2} - L_n \geq 2s$ then 1 is incomparable with $i_1, i_2, \ldots, i_s$ or n is incomparable with $i_k, i_{k-1}, \ldots, i_{k-s+1}$. Since the existence of these edges is unconditioned by the value of H'_{n-2}, (3b) follows.

Let now $c_n = E(L_n)$. We observe that

$$c_{m+n} \geqslant c_m + c_{n+1} \geqslant c_m + c_n \quad \text{(since } c_n \text{ is clearly a monotone sequence).} \tag{4}$$

This is because we obtain a $1-(m+n)$ chain by concatenation of a longest $1-m$ chain with a longest $m-(m+n)$ chain. It is well known that (4) and $c_n \leqslant n$ implies the existence of a real constant c such that

$$\lim_{n\to\infty} \left(\frac{c_n}{n}\right) = c. \tag{5}$$

(see for example Polya and Szëgo [9] Problem 98, p. 23)
Now (2) and (3) imply that

$$-8 = -2 \sum_{s=1}^{\infty} (\tfrac{1}{2})^{s-1} s \leqslant E(L_n) - E(H_{n-2}) \leqslant 2$$

and so, by (5),

$$\lim_{n\to\infty} \left(\frac{E(H_n)}{n}\right) = c. \tag{6}$$

Our previous analysis yields

$$.5654 < c < .610.$$

We now apply Lemma 1 to H_n. We let $X_i = \{j < i : ji \in E(G)\}$, $i = 1, 2, \ldots, n$. We show below that

$$\delta_i \leqslant 3 \log_2 n. \tag{7}$$

It follows that

$$\Pr(|H_n - cn| \geqslant \varepsilon n) = O(\exp(-\varepsilon^2 n/72(\log n)^2)).$$

(for n sufficiently large, $|E(H_n) - cn| \leq (1/2)\varepsilon n$, and we can then apply the lemma).

Let i and $X^{(i)}$ be given. Since

$$E(H_n \mid X^{(i-1)}) = \sum_{Y \subseteq \{1,2,\ldots,i-1\}} E(H_n \mid X^{(i-1)}, X_i = Y) \Pr(X_i = Y).$$

We can deduce (7) from

$$|E(H_n \mid X^{(i-1)}, X_i = Y) - E(H_n \mid X^{(i-1)}, X_i = Z)| \leqslant 3 \log_2 n \tag{8}$$

for all Y, Z.

Let H'_n denote the longest chain in G that does not contain i. Then

$$H'_n(X^{(i-1)}, Y, X_{i+1}, \ldots, X_n) = H'_n(X^{(i-1)}, Z, X_{i+1}, \ldots, X_n)$$

for all $Y, Z, X_{i+1}, \ldots, X_n$. Hence (8) follows from

$$E(H_n - H'_n \mid X^{(i)}) \leqslant 3 \log_2 n, \quad i = i, 2, \ldots, n.$$

This in turn follows from

$$\Pr(H_n - H'_n \geqslant 2 \log_2 n \mid X^{(i)}) \leqslant \left(\frac{e}{\log_2 n}\right)^{\log_2 n}. \tag{9}$$

Suppose that $H_n - H'_n \geqslant 2 \log 2_n$. Let C be a chain of length H_n and $C_1 = C \cap \{1, 2, \ldots, i-1\}$, and $C_2 = C \cap \{i+1, i+2, \ldots, n\}$. Now $H_n - H'_n \leqslant \min\{|C_1|, |C_2|\} + 1$, and so $|C_1|, |C_2| \geqslant \log_2 n - 1$. Next let C'_1 consist of the $[\log_2 n]$ largest elements in C_1, and C'_2 consist of the $[\log_2 n]$ smallest elements in C_2. There can be no $C'_1 - C'_2$ edge in G. Hence, where $k = [\log_2 n]$.

$$\Pr(H_n - H'_n \geqslant 2 \log_2 n \mid X^{(i)}) \leqslant \binom{n-i}{k}\left(\frac{1}{2}\right)^{k^2} \leqslant \left(\frac{ne}{2^k k}\right)^2$$

and (9) follows.

3. Set-Up Number

The set-up number of a linear extension L of a partial order P is the number of pairs of successive elements in L which are incomparable in P. The set-up number of P itself is the minimum over all linear extensions of P of their set-up numbers. If P represents a set of precedence constraints for a series of tasks, then the set-up number is the smallest number of times a single machine processing these tasks must perform a pair of incomparable tasks in succession. In some models this incurs a 'set-up cost', hence the terminology.

In [6], it is shown that the rank of the incidence matrix of P over $\mathbf{Z}_2$ can be used to provide a lower bound on the set-up number. In fact, it requires only a minor modification of their proof to see that this bound applies when the matrix used is the incidence matrix of any relation whose transitive closure is the order relation on P. In our model, the relevant matrix is a random strictly upper triangular 0-1 matrix (where by random we mean that the potential non-zero entries are independently set to either 0 or 1 with probability $\frac{1}{2}$).

Unfortunately, the expected rank of such a matrix is roughly $n - 4 \log_2 n$, and this gives us a trivial lower bound for the set-up number. However the following analysis, similar in spirit to that for height does give a pair of bounds for the expectation s_n of the set-up number S_n, of the form $c_1 n < s_n < c_2 n$, but this time $c_2 - c_1$ is fairly large.

To do this we first require estimates of

$$p_k = \Pr(\text{there exist exactly } k \text{ maximals}).$$

We can find an exact formula for p_k by noting that by symmetry we can replace the word maximal by minimal. First observe that for $1 \leqslant j \leqslant n$,

$$\Pr(j \text{ is minimal}) = (\tfrac{1}{2})^{j-1},$$

and that these events are independent. Therefore, if we let

$$c_n = \prod_{j=1}^{n-1} \left(1 - \frac{1}{2^j}\right)$$

we obtain

$$p_k = c_n \sum_{2 \leqslant n_2 < \cdots < n_k \leqslant n} \prod_{j=2}^{k} \left(\frac{1}{2_j^{n-1} - 1}\right).$$

These values converge very quickly as n grows, and we get the following estimates:

$$p_1 = .2888,$$
$$p_2 = .4640,$$
$$p_3 = .2085,$$
$$p_4 = .0359,$$
$$p_5 = .0027,$$
$$p_6 = .0003.$$

We can now produce an upper bound for the expected set-up number by noting that when we add a new vertex, if it lies above any of the existing maximals, then the set-up number does increase. If we assume that in all other cases, the set-up number does increase, then the average increment in the set-up number will be the probability that a new vertex is incomparable with all the existing maximals. In the case where $p = \frac{1}{2}$, this obviously the same as the probability of having exactly one maximal (sine that requires that the new vertex be comparable to all the existing maximals), so if we denote the expected set-up number by s_n, then for sufficiently large n,

$$\frac{s_n}{n} \leqslant .289.$$

Note that this upper bound is effective in the sense that we can iteratively construct a linear extension with no more than $(.289)n$ expected set-ups by adjoining each new element according to the criteria above.

We obtain a lower bound by considering the addition of m new elements. If exactly j of these lie above all the previous existing maximals, and are mutually incomparable with one another then we must add at least $(j-1)$ new set-ups. This means that we must expect to add at least:

$$a(m) = \sum_{k=1}^{n} \sum_{j=2}^{m} \binom{m}{j} \frac{(j-1)}{2^{jk}} \left(1 - \frac{1}{2^k}\right)^{m-j} \frac{1}{2^{j(j-1)/2}} p_k$$

new set-ups. Since there are m possible additional set-ups, we need to look at the ratios $a(m)/m$). This is maximized when $m = 3$, and $a(3)/3 = .034$.

So we may conclude that, for sufficiently large n,

$$(.034)n \leqslant s_n \leqslant (.289)n.$$

We will now prove a sharp concentration result for the set-up number S_n.

THEOREM 3. *There exists a constant* σ, *with* $.034 \leqslant \sigma \leqslant .289$ *such that*

(a) *if* $\varepsilon > 0$ *is fixed and small, then* $\Pr(|S_n - \sigma n| \geqslant \varepsilon n) = O\left(\exp\left(-\frac{\varepsilon^2 n^{(1/3)}}{16}\right)\right)$,

(b) $\Pr\left(\lim_{n\to\infty} \frac{S_n}{n} = \sigma\right) = 1.$

Proof. Clearly we need only prove (a). First note that

$$s_{m+n} \leqslant s_m + s_n + 1.$$

We can see this by considering the concatenation of the optimal set-up ordering for $\{1, 2, \ldots, m\}$ and that of $\{m+1, m+2, \ldots, m+n\}$. This suffices to show that there exists σ such that

$$\lim_{n\to\infty} \frac{s_n}{n} = \sigma.$$

The bounds for σ follow from our previous analysis. We now wish to apply Lemma 1. Proceeding as in Theorem 2 we would like to show (see (8)) that for a given i, $X^{(i-1)}$,

$$|E(S_n \mid X^{(i-1)}, X_i = Y) - E(S_n \mid X^{(i-1)}, X_i = Z)| \leqslant \text{something small},$$

for all i, $X^{(i-1)}$, Y, Z.

We cannot quite manage this. Instead, let G' be obtained from G by adjoining, where necessary, all edges of the form kl where $l - k > n^{1/3}$. Let Σ denote the event $\{G'$ induces a differential partial order to $G\}$. Then

$$\Pr(|S_n - \sigma n| \geqslant \varepsilon n) \leqslant \Pr(\Sigma) + \Pr(|S_n' - \sigma n| \geqslant \varepsilon n). \tag{10}$$

Now if Σ occurs then for some $k < 1 - n^{1/3}$ there is no 3 element chain in G of the form k, p, l where $k < p < l$. Hence

$$\Pr(\Sigma) \leqslant n^2(\tfrac{3}{4})^{n^{1/3}}. \tag{11}$$

We can deduce immediately that

$$\lim_{n\to\infty} \frac{E(S_n')}{n} = \sigma. \tag{12}$$

We will show later that

$$|S_n'(X^{(i-1)}, Y, X_{i+1}, \ldots, X_n) - S_n'(X^{(i-1)}, Z, X_{i+1}, \ldots, X_n)| \leqslant 2n^{1/3} \tag{13}$$

for all $X^{(i)}, Y, Z, X_{i+1}, \ldots, X_n$. Hence

$$|E(S'_n \mid X^{(i-1)}, X_i = Y) - E(S'_n \mid X^{(i-1)}, X_i = Z)| \leqslant 2n^{1/3} \tag{14}$$

and so

$$\Pr(|S'_n - E(S'_n)| \geqslant \varepsilon n) \leqslant 2\exp\left(-\frac{\varepsilon^2 n}{8n^{2/3}}\right). \tag{15}$$

Our theorem will now follow from (11), (13), and (15). So it remains only to establish that (13) is true.

To this end, let G'' denote the subgraph of G' induced by $\{1, 2, \ldots i - \lfloor n^{1/3} \rfloor, i + \lfloor n^{1/3} \rfloor, i + \lfloor n^{1/3} \rfloor + 1, \ldots, n\}$. Let $S'_Y = (X^{(i-1)}, Y, X_{i+1}, \ldots, X_n)$ and let S'_Z, S''_Y, S''_Z be defined similarly. Observe that

$$S''_Y = S''_Z, \tag{16}$$

since G'' does not depend on X_i.

Now, any linear extension of G'' can be extended to a linear extension of G' since if $x \leqslant i - \lfloor n^{1/3} \rfloor$ and $y \geqslant i + \lfloor n^{1/3} \rfloor$ then y follows x in any linear extension of G'', and so the elements $i - \lfloor n^{1/3} \rfloor + 1, i - \lfloor n^{1/3} \rfloor + 2, \ldots, i + \lfloor n^{1/3} \rfloor - 1$ can be added in the obvious place, and in this order. This extension introduces, at most, $2n^{1/3}$ new set-ups. Thus

$$S'_Y \leqslant S''_Y + 2n^{1/3} \quad \text{and} \quad S'_Z \leqslant S''_Z + 2n^{1/3}. \tag{17}$$

On the other hand, any extension of G' restricts to one of G'' with no extra set-ups, and so

$$S'_Y \geqslant S''_Y \quad \text{and} \quad S'_Z \geqslant S''_Z. \tag{18}$$

Combining (16), (17), and (18) now yields (13).

4. Dimension and Width

In [2] Barak and Erdös prove that the width of P_n is concentrated precisely at the value

$$K_n = \left\lfloor \sqrt{\frac{2\log n}{\log 2} + \frac{1}{4}} + \frac{1}{2} \right\rfloor.$$

The proof rests on some very clever refinements of the observation that if Q_n is an ordered set of size $(1-\varepsilon)\sqrt{2\log_2 n}$, then with probability tending to 1 as n tends to infinity, P_n will contain a subset order isomorphic to Q_n. Obviously this observation can also be applied to the dimension of P_n to get

$$\lim_{n\to\infty} \Pr\left(\dim(P_n) \geqslant \frac{(1-\varepsilon)}{2}\sqrt{\frac{2\log n}{\log 2}}\right) = 1.$$

(since there is an ordered set with $2d$ elements of dimension d), and as the dimension is always less than or equal to the width, the Barak and Erdös result also gives us an upper bound differing from this by a factor of two.

We have been unable to substantially improve these estimates but pose:

CONJECTURE. *The dimension D_n of P_n is sharply concentrated, with expected value approximately*

$$\sqrt{\frac{2\log n}{\log 2}}.$$

5. First Order Properties

From a logical point of view, one of the most interesting properties of random graphs is the *0-1 law*. Very simply put, this states that for any first order sentence σ in the language of graphs,

$$\lim_{n\to\infty} \Pr(G_n \text{ satisfies } \sigma) \in \{0, 1\}.$$

The result was proven by Fagin in [4]. Compton ([3]) has shown that this result also holds for the class of all finite orders if the successive probability distributions are taken to be uniform on the orders of size n.

In fact, the collection of sentences true of 'almost all' random graphs is the theory of the model completion of the class of finite graphs. In [11], Winkler proves that, although his random orders do not satisfy a 0-1 law, it is nonetheless the case that the set of sentences which are true with probability 1 form a model complete theory. This is also true in Compton's framework.

For our random orders however, no 0-1 law holds. For example the probability that there exists a unique minimal element in a random graph order of size n tends to a nonzero limit (approximately .2888) as n tends to infinity. The following facts (all of which can be expressed as first order sentences) do have probability one in the limit:

(1) for any fixed finite ordered set P there exists a set of elements order isomorphic to P;
(2) for any collection of elements of fixed finite size, there exists an element comparable with all of them;
(3) there exists an element comparable to every element;
(4) for any fixed k, there exist k elements satisfying exactly the same strict order relations.

Note that (4) is a particularly strong validation of the conjecture that every partially ordered set P contains a pair of elements a and b such that the proportion of linear extensions of P in which $a < b$ lies between $\frac{1}{3}$ and $\frac{2}{3}$. In fact, it is a consequence of (4) that with probability tending to 1, there will be k elements in a random graph order, for each of whose $k!$ possible arrangements occur equally often in linear extensions of the random graph order. Related to the first order properties of random graph orders is the following question:

Question. For each first order sentence σ does $\lim_{n\to\infty} \Pr(P_n$ satisfies $\sigma)$ exist?

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