Packing Hamilton Cycles in Random and Pseudo-Random Hypergraphs

Alan Frieze* Michael Krivelevich†

February 16, 2011

Abstract

We say that a k-uniform hypergraph C is a Hamilton cycle of type ℓ, for some $1 \leq \ell \leq k$, if there exists a cyclic ordering of the vertices of C such that every edge consists of k consecutive vertices and for every pair of consecutive edges E_{i-1}, E_i in C (in the natural ordering of the edges) we have $|E_{i-1} \setminus E_i| = \ell$. We prove that for $k/2 < \ell \leq k$, with high probability almost all edges of the random k-uniform hypergraph $H(n,p,k)$ with $p(n) \gg \log^2 n/n$ can be decomposed into edge-disjoint type ℓ Hamilton cycles. A slightly weaker result is given for $\ell = k/2$. We also provide sufficient conditions for decomposing almost all edges of a pseudo-random k-uniform hypergraph into type ℓ Hamilton cycles, for $k/2 \leq \ell \leq k$. For the case $\ell = k$ these results show that almost all edges of corresponding random and pseudo-random hypergraphs can be packed with disjoint perfect matchings.

1 Introduction

The subject of Hamilton graphs and Hamiltonicity-related problems is undoubtedly one of the most central in Graph Theory, with a great many deep and beautiful results obtained. Hamiltonicity problems occupy a place of honor in the theory of random graphs too, the reader can consult the monographs of Bollobás [4] and of Janson, Łuczak and Ruciński [14] for an account of some of the most important results related to Hamilton cycles in random graphs. Of particular relevance to the current work is a previous result of the authors [9] who proved that for edge probability $p = p(n) \geq n^{-\epsilon}$ for some constant $\epsilon > 0$, **whp**[1] almost all

*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, U.S.A. Email: alan@random.math.cmu.edu. Supported in part by NSF grant DMS-0753472.

†School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: krivelev@post.tau.ac.il. Research supported in part by a USA-Israel BSF grant and by a grant from the Israel Science Foundation.

[1]An event $\mathcal{E}_n$ occurs *with high probability*, or **whp** for brevity, if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

edges of the random graph $G(n,p)$ can be packed with edge-disjoint Hamilton cycles.

Quite a few results about Hamiltonicity of pseudo-random graphs are available too. Informally, a graph $G = (V, \mathcal{E})$ with $|V| = n$ vertices and $|\mathcal{E}| = m$ edges is pseudo-random if its edge distribution is similar, in some well-defined quantitative way, to that of a truly random graph $G(n,p)$ with the same expected density $p = m\binom{n}{2}^{-1}$. A thorough discussion about pseudo-random graphs, their alternative definitions and properties can be found in the survey [17]. It is well-known that pseudo-randomness of graphs can be guaranteed by imposing conditions on vertex degrees and co-degrees (see, e.g., [21], [5]); we will adopt a similar approach later in the paper when discussing pseudo-random hypergraphs. There are known sufficient criteria for Hamiltonicity in pseudo-random graphs. Also, the above mentioned result of [9] can be extended to the pseudo-random case as well. Since we will employ this result in our arguments, let us state it here formally. A graph G on vertex set $[n]$ is (α, ϵ)-*regular* if

Q_a: $\delta(G) \geq (\alpha - \epsilon)n$, where $\delta(G)$ denotes the minimum degree of G.

Q_b: If S, T are disjoint subsets of $[n]$ and $|S|, |T| \geq \epsilon n$ then $\left|\frac{e_G(S,T)}{|S|\,|T|} - \alpha\right| \leq \epsilon$, where $e_G(S,T)$ is the number of $S - T$ edges in G.

The following is implied by the main theorem of [9]:

Theorem 1 *Let G be an (α, ϵ)-regular graph with n vertices where*

$$\alpha \gg \epsilon \text{ and } \alpha\epsilon^3 \gg \frac{1}{(n \log n)^{1/2}}. \tag{1}$$

Then G contains at least $(\alpha/2 - 4\epsilon)n$ edge-disjoint Hamilton cycles.

Remark 1 *Theorem 2 of [9] only claims to be true for α constant. This was an unfortunate over-cautious statement. The real condition should be the one given in the above theorem. We will justify this claim in the appendix.*

In contrast, much less is known about Hamiltonicity in hypergraphs in general and in random and pseudo-random hypergraphs in particular. Formally, a *hypergraph* H is an ordered pair $H = (V, \mathcal{E})$, where V is a set of vertices, and $\mathcal{E}$ is a family of distinct subsets of V, called edges. A hypergraph H is k-*uniform* if all edges of H are of size k. It is generally believed that k-uniform hypergraphs for $k \geq 3$ are much more complicated objects of study than graphs (corresponding to $k = 2$). Specifically for Hamiltonicity, even extending the definition of a Hamilton cycle in graphs to the case of (uniform) hypergraphs is not a straightforward task. In fact, several alternative definitions are possible. In this paper (in some departure from a relatively standard notation) we will use the following definition. Denote

$$\nu_i = \frac{n}{i},\ 1 \leq i \leq k.$$

Suppose that $1 \leq \ell \leq k$. A *type ℓ Hamilton cycle* in a k-uniform hypergraph $H = (V, \mathcal{E})$ on n vertices is a collection of ν_ℓ edges of H such that for some cyclic order of $[n]$ every edge consists of k consecutive vertices and for every pair of consecutive edges E_{i-1}, E_i in C (in the natural ordering of the edges) we have $|E_{i-1} \setminus E_i| = \ell$. Thus, in a type ℓ Hamilton cycle the sets $C_i = E_i \setminus E_{i-1}$, $i = 1, 2, \ldots, \nu_\ell$, are a partition of V into sets of size ℓ. (An obvious necessary condition for the existence of a cycle of type ℓ in a hypergraph on n vertices is that ℓ divides n. We thus always assume, when discussing Hamilton cycles of type ℓ, that this necessary condition is fulfilled.) In the literature, when $\ell = 1$ we have a *tight* Hamilton cycle and when $\ell = k - 1$ we have a *loose* Hamilton cycle. In the extreme case $\ell = k$ the notion reduces to that of a perfect matching in a hypergraph.

Several recent papers (see, e.g., [13], [16], [18], [20]) provided sufficient conditions for the existence of a type ℓ Hamilton cycle in a k-uniform hypergraph H on n vertices in terms of the minimum number of edges of H passing through any subset of $k - 1$ vertices, thus extending the classical Dirac sufficient condition for graph Hamiltonicity to the hypergraph case. These results however appear to be of rather limited relevance to the current paper, as here we are mostly concerned with sparse hypergraphs (with $o(|V|^k)$ edges), while the above mentioned results are for the (very) dense case.

The main goal of this paper at large is to study Hamiltonicity in random and pseudo-random hypergraphs. A *random k-uniform hypergraph* $H(n, p, k)$ is a hypergraph with vertex set $\{1, \ldots, n\} = [n]$, where each k-tuple of $[n]$ is an edge of the hypergraph independently with probability $p = p(n)$. For the case $k = 2$ the model $H(n, p, k)$ reduces to the classical binomial random graph $G(n, p)$. Up until recently, essentially nothing was known about Hamilton cycles in random hypergraphs. Even the most basic question of the threshold for the appearance of a cycle of type ℓ in $H(n, p, k)$ had not been addressed in the literature. A recent series of papers, see Frieze [8], Dudek and Frieze [6], [7], establish the thresholds for the existence of type ℓ Hamilton cycles, $1 \leq \ell \leq k-1$ up to constant factors or up to arbitrarily slow growing functions of n. In the case $\ell = k$, i.e., the case of perfect matchings – a recent striking result of Johannson, Kahn and Vu [15], has established the order of magnitude of the threshold for the appearance of a perfect matching in a k-uniform random hypergraph.

In this paper, rather than studying the conditions for the existence of a single Hamilton cycle, we study the conditions for the existence of a *packing of almost all edges* of a random or a pseudo-random hypergraph into Hamilton cycles. For $\ell \geq k/2$ we manage to obtain non-trivial results in this direction. It appears that the cases of small ℓ (where adjacent edges along the Hamilton cycle have larger intersection) are harder.

Our first result is about packing Hamilton cycles in random hypergraphs.

Theorem 2 *Suppose that $k/2 < \ell \leq k$ and suppose that $np/\log^2 n \to \infty$. Then* **whp** *$H = H(n, p, k)$ contains a collection of $(1 - \epsilon)\binom{n}{k}p/\nu_\ell$ edge-disjoint type ℓ Hamilton cycles, $\epsilon^2 = \Theta(\log n/(np)^{1/2}) = o(1)$. When $\ell = k/2$ we have the same conclusion for $\epsilon^{11} \gg \log^{7/2} n/(n^{1/2}p^4)$.*

Note that for the case $\ell = k$ the above theorem provides a sufficient condition on the edge probability $p(n)$ for being able to pack **whp** almost all edges of $H(n, p, k)$ into perfect matchings.

Other results of the paper are about packing Hamilton cycles in pseudo-random hypergraphs. For most part, we state the condition of pseudo-randomness of a hypergraph in terms of the number of edges through subsets of vertices of fixed size. These conditions are suggested by the expected numbers of such edges in truly random hypergraphs of the same edge density and are easily seen to hold **whp** in random hypergraphs. Thus our results about pseudo-random hypergraphs are applicable to truly random instances as well. Naturally, the direct approach of Theorem 2 provides a better lower bound on the edge probability $p(n)$.

In this paper we are only able to deal with the case where $\ell \geq k/2$. Let $H = ([n], \mathcal{E})$ be a k-uniform hypergraph with vertex set $[n]$ and m edges. Its density $p = m/\binom{n}{k}$. For a set $X \subseteq [n]$ with $|X| = a < k$ we define its neighbourhood $N_H(X) = \left\{ Y \in \binom{[n]}{k-a} : X \cup Y \in \mathcal{E} \right\}$ and its degree $d_H(X) = |N_H(X)|$.

We first consider $k/2 < \ell < k$ and list the following properties. The value ϵ will be a parameter of *regularity*. P_a says that the "degrees" of small sets are close to being regular; P_b says that the "co-degrees" of small sets are not too large.

$$P_a(s)\textbf{:}\ \min_{S \in \binom{[n]}{s}} d_H(S) = (1 \pm \epsilon) \binom{n}{k-s} p \quad \text{and } P_a = \bigcap_{1 \leq s \leq k-1} P_a(s).$$

$$P_b(s,t)\textbf{:}\ \max_{\substack{S_1, S_2 \in \binom{[n]}{s} \\ |S_1 \cap S_2| = t}} |N_H(S_1) \cap N_H(S_2)| \leq (1+\epsilon) \binom{n}{k-s} p^2 \text{ and } \ P_b = \bigcap_{\substack{1 \leq s \leq k-1 \\ 0 \leq t \leq s}} P_b(s,t).$$

(the notation $A = (1 \pm \epsilon) B$ as a shorthand for the pair of inequalities $(1-\epsilon)B \leq A \leq (1+\epsilon)B$.)

Theorem 3 *Let $H = ([n], \mathcal{E})$ be a k-uniform hypergraph with m edges and let $p = m/\binom{n}{k}$. Suppose that $k/2 < \ell < k$ and $1 > \epsilon^5 \gg \log^3 n/(n^{1/2} p^2)$ and suppose that H satisfies properties $\mathcal{P} = \{P_a, P_b\}$. Then H contains a collection of $(1 - 2\epsilon^{1/3}) m/\nu_\ell$ edge-disjoint type ℓ Hamilton cycles.*

The restriction $1 > \epsilon$ is for relevance and the restriction $\epsilon^5 \gg \log^3 n/(n^{1/2} p^2)$ is used in the proof (see Lemma 7).[2] The latter condition can be relaxed a little through a more careful implementation of our argument.

We do not really need $P_a, P_b(s,t)$ to hold for all s, t but using the above simplifies the statement of the theorem. The proof will expose the actual values of s, t for which these properties are needed. Also, one can observe that in fact, for example, the condition P_a, which is a

[2]We use the notation $a_n \gg b_n$ as shorthand for $a_n/b_n \to \infty$ as $n \to \infty$.

conjunction of the conditions $P_a(s)$ for different s, is essentially implied by the condition $P_a(k-1)$ alone. In any case any reasonable definition of a pseudo-random hypergraph is likely to yield all such conditions. This is certainly the case for random hypergraphs.

When $\ell = k/2$ we will use the result from [9] as our main technical tool, and the above stated definition of (α, ϵ)-regular graphs. Here the definition of a pseudo-random hypergraph is explicitly tailored to our application. Let $H = (V, \mathcal{E})$ be a k-uniform hypergraph with vertex set $V = [n]$. Let $\mathcal{P} = (X_1, X_2, \ldots, X_{\nu_\ell})$ be a partition of $[n]$ into ν_ℓ parts each of size ℓ. The graph $G_{\mathcal{P}} = G_{\mathcal{P}}(H)$ has vertex set $[\nu_\ell]$ and an edge (i, j) whenever $E = X_i \cup X_j \in \mathcal{E}(H)$. We now say that H is (p, ϵ)-*regular* if for a *randomly chosen* $\mathcal{P}$, the graph $G_{\mathcal{P}}$ is (p, ϵ)-regular **qs**[3].

Theorem 4 *Let $H = ([n], \mathcal{E})$ be a k-uniform hypergraph with m edges and let $p = m/\binom{n}{k}$. Suppose that H is (p, ϵ)-regular k-uniform hypergraph with $\ell = k/2$ and*

$$\epsilon^{11/4} p n^{1/8} \gg \log^{7/8} n.$$

Then H contains a collection of $(1 - 20\epsilon)m/\nu_\ell$ edge-disjoint type ℓ Hamilton cycles.

We finally consider the case $k = \ell$. Here we will be packing perfect matchings as opposed to Hamilton cycles.

Theorem 5 *Let $H = ([n], \mathcal{E})$ be a k-uniform hypergraph with m edges that satisfies P_a, P_b and suppose that $1 \gg \epsilon \gg \log^5 n/(n^{1/2}p^2)$. Then H contains a collection of $(1 - 4\epsilon^{1/3})m/\nu_k$ edge-disjoint perfect matchings.*

These are the first results of any significance on packing Hamilton cycles in random and pseudo-random hypergraphs. We have no reason to believe that they are tight and it would be interesting to sharpen them.

The case $\ell < k/2$ is more difficult. Frieze, Krivelevich and Loh [10] have analysed the case $k = 3, \ell = 1$ and proved a result of the same flavour as the above. This has since been extended to more general k, ℓ in Bal and Frieze [2].

An interesting point of reference for our theorems is results about *perfect* decompositions of the edge set of a complete k-uniform hypergraph K_n^k into Hamilton cycles of various types (assuming of course some natural divisibility conditions). These include a recent result of Bailey and Stevens [1] about packing tight Hamilton cycles and a famous result of Baranyai [3] about decomposing the edge set of K_n^k into perfect matchings. While we do not – and can not for obvious reasons – achieve perfect decomposition, but rather pack almost all edges, our results apply to a wide class of hypergraphs, including relatively sparse hypergraphs.

In the next section we cite and prove several general tools applied later in the proofs. In Section 3 we focus on $H = H(n, p, k)$ and first prove Theorem 2 for $k = 3$. We then provide a

[3]An event $\mathcal{E}_n$ occurs *quite surely*, or **qs** for brevity, if $\mathbf{Pr}(\mathcal{E}_n) = 1 - O(n^{-C})$ for any positive constant C.

general scheme of the proof, applicable to both the random and the pseudo-random cases. In Section 5 we give a proof of the random case for general k, ℓ. In Section 6 we prove Theorems 3, 4 and 5. The last section is devoted to concluding remarks.

2 Tools

We start with stating very standard Chernoff-type bounds for binomial random variables: for $0 \leq \epsilon \leq 1$:

$$\mathbf{Pr}(Bin(m,\xi) - m\xi \leq -\epsilon m\xi) \leq e^{-\epsilon^2 m\xi/2} \tag{2}$$

$$\mathbf{Pr}(Bin(m,\xi) - m\xi \geq \epsilon m\xi) \leq e^{-\epsilon^2 m\xi/3} \tag{3}$$

Next, we give two statements about the (high probability) existence of many disjoint perfect matchings in random and in pseudo-random bipartite graphs. For the (somewhat easier) truly random case we will need the obtained estimate on the number of matchings to hold with probability polynomially close to 1.

Lemma 1 *Let $k \geq 1$ be an integer. Let G be a random bipartite graph with sides A, B of size $|A| = |B| = n$, where each pair (a, b), $a \in A, b \in B$, is an edge of G independently and with probability at least p. Assume $p(n) \gg \log n/n$. Then with probability $1 - o(n^{-k})$, G contains a family of $(1-\delta)np$ edge disjoint perfect matchings, where*

$$\delta = \left(\frac{4(k+3)\log n}{np}\right)^{1/2} = o(1) \ .$$

Proof Due to monotonicity is is enough to prove the lemma for the case when all edge probabilities are exactly p. Set $n_0 = (1-\delta)np$. The Max-Flow Min-Cut theorem tells us that the following is a necessary and sufficient condition for G to have n_0 edge-disjoint perfect matchings: Suppose that we make up a network with source σ and sink τ and join σ to each vertex of A by an edge of capacity n_0 and each vertex of B to τ by an edge of capacity n_0. Each edge of G is given capacity one. Suppose that our minimum cut is $X : \bar{X}$ and $S = A \cap X$ and $T = B \cap X$ then a necessary and sufficient condition for the existence of n_0 disjoint perfect matchings is that

$$(n - |S|)n_0 + |T|n_0 + e(S, B \setminus T) \geq n_0 n \,,$$

which reduces to

$$m \geq (s-t)n_0 \tag{4}$$

for all $S \subseteq A, T \subseteq B$, if $|S| = s, |T| = t, m = e(S, B \setminus T)$.

Note that we need only verify (4) computationally for $t \leq n/2$. When $t > n/2$ we could repeat our computations to show that with the required probability $e(B \setminus T, A \setminus (A \setminus S)) \geq ((n-t) - (n-s))n_0$.

We have:

$$Pr(\exists S \subseteq A, T \subseteq B, |S| > |T|, |T| \le n/2 : \; e(S, B \setminus T) \le (1-\delta)|S|(n-|T|)p) \le$$
$$\sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \binom{n}{s}\binom{n}{t} \exp\left\{-\frac{\delta^2}{2} s(n-t)p\right\} \le$$
$$\sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \binom{n}{s}\binom{n}{t} \exp\left\{-\delta^2 snp/4\right\}. \tag{5}$$

Assume first that $\binom{n}{s} \ge \binom{n}{t}$. Then

$$(5) \le \sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \left(\frac{n^2e^2}{4s^2} \cdot e^{-\delta^2 np/4}\right)^s = \sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \left(\frac{n^2e^2}{4s^2} \cdot n^{-k-3}\right)^s = o(n^{-k}).$$

When $\binom{n}{s} \le \binom{n}{t}$ we can replace (5) by

$$\sum_{s=n/2}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \left(\frac{n^2e^2}{4t^2} \cdot n^{-k-3)}\right)^t = o(n^{-k}).$$

Here we have used $s \ge t$.

It follows (see (4)) that with the required probability simultaneously for all relevant sets S, T,

$$m \ge (1-\delta)s(n-t)p \ge (1-\delta)(s-t)np = (s-t)n_0\,.$$

□

In the next lemma we prove the existence of many disjoint perfect matchings in a bipartite pseudo-random graphs with equal sides, where pseudo-randomness is given through/controlled by vertex degrees and co-degrees.

Lemma 2 *Let G be a bipartite graph with vertex set $A \cup B$ where $|A| = |B| = n$. Suppose that the minimum degree in G is at least $(1-\theta)dn$ and the maximum co-degree of two vertices is at most $(1+\theta)d^2n$ for some small value $\theta \ll 1$. Suppose further that $\theta^{4/3}d^2n \gg 1$. Then G contains a collection of $(1-\theta^{1/3})dn$ edge-disjoint perfect matchings.*

The assumption $\theta^{4/3}d^2n \gg 1$ in the above lemma is mostly for convenience and will be good enough for our purposes; it can be relaxed somewhat through a more careful analysis.

Proof Let $d_0 = (1-\theta)d$ and $d_1 = (1-\theta^{1/3})d$. Going back to (4) we see that we need to show that

$$m \ge (s-t)d_1n \tag{6}$$

for all $S \subseteq A, T \subseteq B$, $|S| = s, |T| = t, m = e(S, B \setminus T)$. Obviously we can assume $s > t$.

Now,

$$m \geq s(d_0 n - t)$$

and so (6) is satisfied if

$$s(d_0 n - t) \geq (s - t) d_1 n$$

or

$$s(d_0 - d_1)n \geq (s - d_1 n)t.$$

In particular, (6) holds if $t \leq (d_0 - d_1)n$. Furthermore, we also have

$$m \geq (n - t)(d_0 n - (n - s)).$$

If $n - t \leq (d_0 - d_1)n$ then this implies that (6) holds.

So we assume from now on that

$$(d_0 - d_1)n < \min\{t, n - t\}. \tag{7}$$

We can further assume that $t \leq n/2$. For $t > n/2$ we can reverse the roles of A, B and show that $e(B \setminus T, A \setminus (A \setminus S)) \geq ((n - t) - (n - s))d_1 n$, which is (6).

We now perform the usual double counting trick by estimating the number of paths of the form S, B, S in two ways. On one hand, each such path corresponds to a common neighbor of a pair of vertices $a_1, a_2 \in S$. Therefore, the quantity to be estimated is at most $\binom{s}{2} d_2^2 n$, where $d_2 = (1 + \theta)^{1/2} d$. On the other hand, it is exactly

$$\sum_{b \in B} \binom{d(b, S)}{2} = \sum_{b \in B \setminus L} \binom{d(b, S)}{2} + \sum_{b \in L} \binom{d(b, S)}{2}$$

where $d(b, S)$ is the number of neighbors of b in S in the graph G. Since $\sum_{b \in B \setminus L} d(b, S) = m$, we can estimate the first summand as follows:

$$\sum_{b \in B \setminus T} \binom{d(b, S)}{2} \geq (n - t) \binom{\frac{m}{n-t}}{2} = \frac{m \left(\frac{m}{n-t} - 1 \right)}{2}.$$

As for the second summand, the number of edges between T and S can be estimated from below by $d_0 ns - m$, and therefore

$$\sum_{b \in L} \binom{d(b, S)}{2} \geq t \binom{\frac{d_0 sn - m}{t}}{2} = \frac{d_0 sn - m}{2} \left(\frac{d_0 sn - m}{t} - 1 \right).$$

It follows that

$$m \left(\frac{m}{n - t} - 1 \right) + (d_0 sn - m) \left(\frac{d_0 sn - m}{t} - 1 \right) \leq s(s - 1) d_2^2 n.$$

After performing straightforward arithmetic manipulations, we get to:

$$(m - d_0 s(n-t))^2 \leq s(n-t)(std_2^2 - std_0^2 + d_0 t)\,.$$

Recalling the definitions of d_0 and d_2, we see that

$$d_2^2 - d_0^2 = (1 + \theta - (1-\theta)^2)d^2 = (3\theta + \theta^2)d^2\,.$$

Also, since $s \geq t$ and $\ell \geq (d_0 - d_1)n \geq \theta^{1/3}dn/2$ by (7), we see that $\theta ds \geq \theta^{4/3}d^2n/2 \gg 1$ by the lemma's assumption. Hence $d_0 t \ll \theta d^2 st$. We thus arrive at the following inequality:

$$(m - d_0 s(n-t))^2 \leq 4\theta d^2 s^2 t(n-t)\ .$$

Since $t \leq n/2$, we have

$$\begin{aligned} m &\geq (n-t)sd_0\left(1 - \frac{2t^{1/2}\theta^{1/2}}{(1-\theta)(n-t)^{1/2}}\right) \\ &\geq (n-t)sd_0\left(1 - \frac{2\theta^{1/2}}{1-\theta}\right). \end{aligned}$$

This implies (6) if

$$\frac{d_1}{d} \leq \frac{(n-t)s}{(s-t)n}\left(1 - \frac{2\theta^{1/2}}{1-\theta}\right)(1-\theta).$$

Since $\frac{s(n-t)}{(s-t)n} \geq 1$, it is enough to verify that

$$\frac{d_1}{d} \leq \left(1 - \theta - 2\theta^{1/2}\right).$$

This is implied by

$$d_1 \leq d(1 - \theta^{1/3}).$$

Thus there will be $d(1-\theta^{1/3})$ edge-disjoint perfect matchings. □

3 Illustrative case: random hypergraphs, $k = 3$, $\ell = 2$

The proof of the general case (at least for $\ell > k/2$) is based on the same idea as for the case $k = 3, \ell = 2$ but is heavier on notation and will be therefore given afterwards. Hopefully, the reader will find it useful to consider the simplest case first.

We will construct the Hamilton cycles via the following algorithm:

A_1: Choose $r = n(np)^{1/2}$ random partitions $(X_i, Y_i), i = 1, 2, \ldots r$, of V into two sets of size ν_2.

We use the notation

$$X_i = \{x_{i,1} < x_{i,2} < \cdots < x_{i,\nu_2}\} \textit{ and } Y_i = \{y_{i,1} < y_{i,2} < \cdots < y_{i,\nu_2}\}.$$

For each i we choose a random permutation σ_i on X_i and define a sequence

$$\Gamma_i = (x_{i,\sigma_i(1)}, x_{i,\sigma_i(2)}, \ldots, x_{i,\sigma_i(\nu_2)}, x_{i,\sigma_i(\nu_2+1)} = x_{i,\sigma_i(1)}).$$

A_2**:** At this point we expose the edges of $H(n,p,3) = ([n], \mathcal{E}_p)$.

A_3**:** Suppose now that for edge $E \in \mathcal{E}_p$ there are $f(E)$ instances i such that for some $1 \le j = j(i) \le \nu_2$, we have $X_i \cap E = \{x_{i,\sigma_i(j)}, x_{i,\sigma_i(j+1)}\}$. If $f(E) > 0$ choose one of these $f(E)$ instances at random and label the edge E with the chosen i; if $f(E) = 0$, the edge E stays unlabelled. Let $H_i \subset H$ be the subhypergraph of all edges labeled by i.

A_4**:** Let G_i be the bipartite graph with vertex set $A_i \cup B_i$ defined as follows: A_i, B_i are disjoint copies of $[\nu_2]$. Add edge (a,b) to G_i if $E = (x_{i,\sigma_i(a)}, y_{i,b}, x_{i,\sigma_i(a+1)}) \in H_i$ (i.e., $(a,b) \in G_i$ if the a-th pair of the sequence Γ_i united with the vertex $y_{i,b}$ forms an edge E of H labeled by i).

A_5**:** We claim that **whp** (see Lemma 3 below) each G_i will contain at least

$$n_0 = (1-\delta)\nu_2 p_0$$

edge-disjoint perfect matchings.

Here

$$\delta = \left(\frac{72n\log n}{r}\right)^{1/2} \text{ and } f_0 = r\rho + (12r\rho\log n)^{1/2} \text{ and } p_0 = \frac{p}{f_0}. \tag{8}$$

We set

$$\rho = \frac{\nu_2^2}{\binom{n}{3}}.$$

Remark 2 *Note that ρ is the probability that instance i is one of the $f(E)$ instances in A_3 for edge $E \in \mathcal{E}$, as we will argue in Lemma 3. Also, we will argue that each graph G_i described above is a truly random bipartite graph with edge probability at least p_0. The parameters are set so as to ensure that $p_0 \gg \log n/n$, thus enabling us to invoke Lemma 1.*

Each such matching gives rise to a loose Hamilton cycle of H_i and these will be edge-disjoint by construction. Indeed suppose that our matching is $(e_a, \phi(e_a)), a = 1, 2, \ldots, \nu_2$, where the edges e_a are ordered according to the order of their appearance along the Hamilton cycle Γ_i. From this we obtain the type 2 Hamilton cycle with edges $E_a = e_a \cup \{\phi(e_a)\}$. Since the subhypergraphs H_i are edge-disjoint and since distinct edges in the graph G_i correspond to distinct edges of H_i, the so obtained Hamilton cycles in H are indeed edge-disjoint.

It follows that **whp** $H(n,p,3)$ contains at least rn_0 edge-disjoint Hamilton cycles. Finally,

$$\begin{aligned} n_0 r &= (1-\delta)\nu_2 p_0 r = (1-\delta)\frac{\nu_2 p}{f_0} r = (1-\delta)\,\frac{\nu_2 pr}{r\rho + (12r\rho\log n)^{1/2}} \\ &= \frac{\nu_2 pr}{r\cdot\frac{\nu_2^2}{\binom{n}{3}}}\,\frac{1-\delta}{1+\left(\frac{12\log n}{r\rho}\right)^{1/2}} \geq \frac{\binom{n}{3}p}{\nu_2}\left(1-\delta-\left(\frac{12\log n}{r\rho}\right)^{1/2}\right) \\ &= \frac{\binom{n}{3}p}{\nu_2}\left(1-\Theta(\log n/(np)^{1/2})\right)\,, \end{aligned}$$

proving Theorem 2 for this case.

Lemma 3

$$\mathbf{Pr}(G_i \textit{ does not contain } n_0 \textit{ edge-disjoint perfect matchings}) = o(n^{-3}).$$

Proof

For a triple $E \subset [n]$, we say that $1 \leq i \leq r$ *includes* E if the set $E \cap X_i$ is of size 2 and is one of the edges of the cycle Γ_i. Thus the random variable $f(E)$ counts the number of partitions (X_i, Y_i) that include E. Observe that the i-th partition includes a fixed triple E with probability

$$\binom{3}{2}\frac{\binom{n-3}{\nu_2-2}}{\binom{n}{\nu_2}}\,\frac{\nu_2}{\binom{\nu_2}{2}} = \frac{3n}{2(n-1)(n-2)} = \frac{\nu_2^2}{\binom{n}{3}} = \rho$$

(first choose two elements of $E\cap X_i$, then choose X_i to intersect E in exactly these two elements, then choose a random permutation σ_i X_i – due to symmetry the probability that $E \cap X_i$ is one of the ν_2 edges it defines is $\nu_2\binom{\nu_2}{2}^{-1}$). Moreover the events "$i$ includes E" are mutually independent for different i. Therefore, the random variable $f(E)$ is distributed binomially with parameters r and ρ. Now using the Chernoff bounds 2, 3, we see that with probability at least $1-o(1)$ we have $1 \leq f(E) \leq f_0$ for all $\binom{n}{3}$ possible edges. So assume that indeed $1 \leq f(E) \leq f_0$ for all E. Moreover, the values of $f(E)$ are determined by Step A_1 of our construction and are thus independent of the appearance of random edges at Step A_2. For $1 \leq a, b \leq \nu_2$, the pair (a,b) is an edge of the random auxiliary graph G_i if the corresponding triple E is an edge of the random hypergraph H and is chosen to be labeled by i. Thus $(a,b) \in E(G_i)$ independently and with probability at least $p/f_0 = p_0$.

Therefore we can **whp** reduce our problem to showing that with probability $1-o(n^{-3})$ the random bipartite graph K_{ν_2,ν_2,p_0} satisfies has a family of n_0 edge disjoint perfect matchings. This is an immediate corollary of Lemma 1. $\square$

4 A general scheme

With a roadmap in mind, we proceed to the general case. There are six cases to deal with (three random and three pseudo-random) and they are all variations on the following scheme: It depends on parameters

$$r, N_X,\ N_Y = n - N_X,\ k_X, k_Y,\ \nu_X = \frac{N_X}{k_X},\ \nu_Y = \frac{N_Y}{k_Y},\ \epsilon,\ f_0.$$

that vary with case.

Define

$$\rho = \rho_{k,\ell} = \frac{\nu_\ell^2}{\binom{n}{k}}\ .$$

General Scheme:

S_1**:** Choose r random partitions $(X_i, Y_i), i = 1, 2, \dots r$, of $[n]$ into two sets of size N_X and N_Y respectively.

We use the notation

$$X_i = \{x_{i,1} < x_{i,2} < \cdots < x_{i,N_X}\}\ \textit{and}\ Y_i = \{y_{i,1} < y_{i,2} < \cdots < y_{i,N_Y}\}\,.$$

S_2**:** For each i we let σ_i be a random permutation of X_i and let τ_i be a random permutation of Y_i. Form the partition $\mathcal{X} = (X_{i,a}, a = 1, 2, \dots, \nu_X)$ of X_i into sets of size k_X and the partition $\mathcal{Y} = (Y_{i,b}, b = 1, 2, \dots, \nu_Y)$ of Y_i into sets of size k_Y. Here

$$X_{i,a} = \left\{x_{i,\sigma_i((a-1)(k-\ell)+1)}, \dots, x_{i,\sigma_i(a(k-\ell))}\right\},\ Y_{i,b} = \left\{y_{i,\tau_i((b-1)(2\ell-k)+1)}, \dots, y_{i,\tau_i(b(2\ell-k))}\right\}.$$

S_3 This part of the scheme is more case specific. Some of the edges $\mathcal{E}$ will be *represented* by an edge of a graph G_i.

(a) $k/2 < \ell < k$**:**

$$k_X = k - \ell,\ k_Y = 2\ell - k,\ N_X = \nu_\ell k_X,\ N_Y = \nu_\ell k_Y.$$

We say that i is a candidate for $E \in \mathcal{E}$ if there exist a, b such that $E = X_{i,a} \cup Y_{i,b} \cup X_{i,a+1}$.

(b) $\ell = k/2$**:**

$$N_X = n,\ N_Y = 0,\ k_X = \ell,\ k_Y = \ell.$$

We say that i is a candidate for $E \in \mathcal{E}$ if there exist a, b such that $E = X_{i,a} \cup X_{i,b}$.

(c) $\ell = k$**:**

$$k_X = \lfloor k/2 \rfloor\,,\ k_Y = \lceil k/2 \rceil\,,\ N_X = \nu_k k_X,\ N_Y = \nu_k k_Y.$$

We say that i is a candidate for $E \in \mathcal{E}$ if there exist a, b such that $E = X_{i,a} \cup Y_{i,b}$.

S_4**:** Suppose now that for $E \in \mathcal{E}$ there are $f(E)$ instances i that are a candidate for E. If $f(E) > 0$ then we choose one of the $f(E)$ instances at random and label edge E with the chosen $i = \psi(E)$. $f(E) = 0$, the edge E stays unlabeled. Let H_i be the subhypergraph of H formed by the edges of H labeled by i.

We will argue in all cases that

$$\mathbf{Pr}((1-\epsilon)f_0 \leq f(E) \leq f_0,\ for\ all\ E \in \mathcal{E}) = 1 - o(1). \tag{9}$$

S_5**:** We now describe a graph G_i that is determined by the edges of H_i. This part of the scheme is also case specific.

(a) $k/2 < \ell < k$**:** G_i is the bipartite graph with vertex partition $A_i \uplus B_i$ comprising disjoint copies of $[\nu_\ell]$. For $a \in A_i$ and $b \in B_i$ we make (a,b) an edge of G_i if $E = X_{i,a} \cup Y_{i,b} \cup X_{i,a+1} \in \mathcal{E}$ and E is labeled with i.

(b) $\ell = k/2$**:** G_i is the graph with vertex set $[\nu_\ell]$, where $a, b \in [\nu_\ell]$ are connected by an edge if $X_{i,a} \cup X_{i,b}$ is an edge of H labeled by i.

(c) $\ell = k$**:** G_i is the bipartite graph with vertex partition $A_i \uplus B_i$ comprising disjoint copies of $[\nu_k]$. For $a \in A_i$ and $b \in B_i$ we make (a,b) an edge of G_i if $E = X_{i,a} \cup X_{i,b} \in \mathcal{E}$ and E is labeled with i.

S_6**:** Let

$$p_0 = p/f_0\,.$$

The construction succeeds:

(a) $k/2 < \ell < k$**:** If for all i, G_i contains $(1-\epsilon)\nu_\ell p_0$ edge disjoint perfect matchings.

(b) $\ell = k/2$**:** If for all i, G_i contains $(1-20\epsilon)r\nu_\ell$ edge-disjoint Hamilton cycles.

(c) $\ell = k$**:** If for all i, G_i contains $(1-\epsilon)\nu_k p_0$ edge disjoint perfect matchings.

We argue now that each matching or cycle in $S_6(a)-(c)$ gives rise to a type ℓ Hamilton cycle in H. We will argue that these Hamilton cycles are edge disjoint. Then it will be a matter of checking that the procedure succeeds **whp** once we have decided on the various parameters and that most edges of H are covered.

(a) $k/2 < \ell < k$**:** A perfect matching of G_i can be expressed as a bijection $f : A_i \to B_i$. This gives us the Hamilton cycle $E_a = X_{i,a} \cup Y_{i,f(a)} \cup X_{i,a+1}$, $a = 1, 2, \ldots, \nu_\ell$ of H. If $a \neq a'$ then $E_a \neq E_{a'}$ and an edge E can appear once only in $G_{\psi(E)}$. Thus the Hamilton cycles produced by the algorithm are edge disjoint.

(b) $\ell = k/2$**:** A Hamilton cycle of G_i can be expressed as a permutation π of $[\nu_\ell]$. This gives us the Hamilton cycle $E_a = X_{i,a} \cup X_{i,\pi(a)}$, $a = 1, 2, \ldots, \nu_\ell$ of H. The argument for Case (a) shows that the Hamilton cycles produced by the algorithm are edge disjoint.

(c) $\ell = k$**:** A perfect matching of G_i gives us the Hamilton cycle $E_a = X_{i,a} \cup Y_{i,f(a)}$, $a = 1, 2, \ldots, \nu_\ell$ of H. The argument for Case (a) shows that the Hamilton cycles produced by the algorithm are edge disjoint.

We will now go through the six cases of the paper, one by one, showing that with carefully chosen values of $r, N_X, k_X, k_Y, \epsilon, f_0$, the above scheme can be shown to work **whp**.

We continue with the analysis of $H(n,p,k)$. In this case we only expose the edges of H after the completion of Step S_2.

5 Random hypergraphs

We now provide a proof of the random case (Theorem 2). It is simpler than the proof for pseudo-random hypergraphs and hopefully will help in the understanding of the proofs of Theorems 3 and 4.

5.1 $k/2 < \ell < k$

Let

$$r = n^{k-2}(np)^{1/2}, \; \epsilon = \left(\frac{4(k+3)k! \log n}{\ell (np)^{1/2}}\right)^{1/2}, \; f_0 = \rho r + (4k\rho r \log n)^{1/2}. \tag{10}$$

Let also

$$n_0 = (1-\epsilon)\nu_l p_0 \,.$$

Lemma 4

$$\mathbf{Pr}(G_i \textit{ does not contain } n_0 \textit{ edge-disjoint perfect matchings}) = o(n^{-k}).$$

Proof The edges of G_i appear independently with probability $p/f(E)$ where $f(E)$ has distribution $Bin(r, \rho)$, $\rho = \rho_{k,\ell} = \frac{\nu_\ell^2}{\binom{n}{k}}$. (To see it, for a fixed partition (X_i, Y_i) and a fixed pair of permutations (σ_i, τ_i) of X_i, Y_i), resp., the index i includes ν_ℓ^2 k-tuples from $[n]$. Therefore by symmetry a random i includes a fixed k-tuple E with probability $\frac{\nu_\ell^2}{\binom{n}{k}}$.) Now $\epsilon^2 r\rho \geq 4k \log n$ and so (2) and (3) imply (9).

We have thus reduced our problem to showing that with probability $1 - o(n^{-k})$ the random bipartite graph $K_{\nu_\ell, \nu_\ell, p_0}$ contains n_0 edge-disjoint perfect matchings. We can use p_0 as probability, because of (9). Applying Lemma 1 gives the desired result. $\square$

We have thus shown that **whp** $H(n,p,k)$ contains at least

$$(1-\epsilon)r\nu_\ell p_0 = \frac{(1-\epsilon)r\nu_\ell p}{\rho_{k,\ell}r + (4k\rho_{k,\ell}r\log n)^{1/2}} \geq (1-2\epsilon)\binom{n}{k}p/\nu_\ell$$

edge disjoint type ℓ Hamilton cycles. This confirms Theorem 2 for this case.

5.2 $\ell = k/2$

Let

$$r = \omega n^{k-2}\epsilon^{-2}\log n,\ \epsilon = \left(\frac{\omega^5\log^{7/2} n}{n^{1/2}p^4}\right)^{1/11},\ f_0 = \rho r + (4k\rho r\log n)^{1/2}$$

where $\rho = \binom{\nu_\ell}{2}/\binom{n}{k}$.

The distribution of $f(E)$ is $Bin(r,\rho)$ and $\epsilon^2 r\rho \gg \log n$ and so (9) holds by the Chernoff bounds.

Lemma 5 *Each G_i is* **qs** $((1-2\epsilon)p_0, 2\epsilon p_0)$*-regular.*

Proof Given (9), the degree of vertex v in G_i dominates $Bin(\nu_\ell-1,p_0)$ and so Property Q_a holds from Chernoff bounds. Observe that $\nu_\ell p_0 = \Omega((np)^{1/2}) \gg \log n$. Similarly the number of edges between two sets S,T dominates $Bin(|S|\,|T|,p_0)$ and is dominated by $Bin(|S|\,|T|,(1-\epsilon)^{-1}p_0)$ and Property Q_b also holds from Chernoff bounds. □

Applying Theorem 1 we see that **whp** H contains at least

$$r\left((1-2\epsilon)p_0/2 - 8\epsilon p_0\right)\nu_\ell \geq \binom{n}{k}\frac{p}{\nu_\ell}(1-20\epsilon)$$

edge-disjoint type $k/2$ Hamilton cycles, completing the proof of Theorem 2 for this case. (It is the second inequality in (1) that determines the value of ϵ here).

5.3 $\ell = k$

Let

$$r = n^{k-2}(np)^{1/2},\ \epsilon = 10k!\left(\frac{4(k+3)k^2\log n}{k!(np)^{1/2}}\right)^{1/2},\ f_0 = \rho r + (4k\rho r\log n)^{1/2} \tag{11}$$

where $\rho = \rho_{k,k}$.

It is left to argue that **whp** each G_i contains at least $(1-\epsilon)\nu_k p_0$ edge-disjoint perfect matchings.

Lemma 6

$$\mathbf{Pr}(G_i \textit{ does not contain } (1-\epsilon)\nu_k p_0 \textit{ edge-disjoint perfect matchings}) = o(n^{-k}).$$

Proof Equation (9) holds, as in the case $k/2 < \ell < k$. The claim follows again from Lemma 1. □

We have thus shown that **whp** $H(n,p,k)$ contains at least

$$(1-\epsilon)r\nu_k p_0 = \frac{(1-\epsilon)r\nu_\ell p}{\rho r + (4k\rho r\log n)^{1/2}} \geq (1-2\epsilon)\binom{n}{k}p/\nu_k$$

edge disjoint type k Hamilton cycles. This confirms Theorem 2 for this case.

6 Pseudo-random hypergraphs

In this section we prove Theorems 3, 4 and 5. We follow the same strategy as described in Section 4. There are complications caused by the notation that we have to add and also by the fact that H is not random.

6.1 $k/2 < \ell < k$ (Theorem 3).

We first choose f_0 such that

$$\frac{\log^2 n}{\epsilon^4} \ll f_0^2 \ll \frac{\epsilon n^{1/2} p^2}{\log n}. \tag{12}$$

We then choose

$$r = \left(1 - \frac{\epsilon}{2}\right)\binom{n}{k}\frac{f_0}{\nu_\ell^2}.$$

$f(E)$ is distributed as $Bin(r, \rho_{k,\ell})$ for each edge of H and the lower bound on f_0 together with the definition of r will then imply that (9) holds.

(The two inequalities in (12) determine our lower bound on ϵ).

We know from Lemma 2 that if we can prove that the degrees and co-degrees of our bipartite graphs G_i "behave", then we can deduce the existence of many disjoint perfect matchings and so get our packing of Hamilton cycles. Given Lemma 2, all we need to do is to estimate the degrees and co-degrees of vertices in a fixed G_i.

Lemma 7 **Whp**, *over our random choices of $X_i, Y_i, \sigma_i, \tau_i$, each G_i has minimum degree at least $(1-2\epsilon)\nu_\ell p_0$ and maximum co-degree of at most $(1+5\epsilon)\nu_\ell p_0^2$.*

Proof We fix i and focus on G_i. We first show that the minimum degree in G_i is large. We first fix $a \in A_i$. The vertex a corresponds to the block $X_{i,a}$ of σ_i. Condition on $X_{i,a} \cup X_{i,a+1} = S$ for some $S \subset [n]$, $|S| = 2(k-\ell)$. We expose a random subset Y_i first. Let Z_a^* be the number of edges $E \in \mathcal{E}$ such that $S \subset E$ and $E \cap Y_i = E - S$. For each edge $E \in N_H(S)$

$$\mathbf{Pr}(E \cap Y_i = E - S) = \frac{\binom{n-k}{N_Y-(2\ell-k)}}{\binom{n-2(k-\ell)}{N_Y}} = \frac{(N_Y)_{2\ell-k}}{(n-2(k-\ell))_{2\ell-k}} = \left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{2\ell-k}{\ell}\right)^{2\ell-k} .$$

Therefore by assumption P_a,

$$\mathbf{E}(Z_a^*) \geq (1-\epsilon)\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{2\ell-k}{\ell}\right)^{2\ell-k}\binom{n-2(k-\ell)}{2\ell-k}p .$$

Since changing the fate of one vertex with respect to the choice of Y_i changes the value of Z_a^* by at most

$$\Delta_a = \max_{S' \in \binom{[n]}{2(k-\ell)+1}} d_H(S') ,$$

and the latter quantity is bounded by $(1+\epsilon)\binom{n}{2\ell-k-1}p$ by assumption P_a, we get by the Azuma-Hoeffding inequality that for any $t > 0$

$$\mathbf{Pr}(Z_a^* \leq \mathbf{E}(Z_a^*) - t) \leq \exp\left\{-\frac{2t^2}{n\Delta_a^2}\right\} . \tag{13}$$

Here we are using the following inequality: Let S_n denote the set of permutations of $[n]$ and let $f : S_n \to \Re$ be such that $|f(\pi) - f(\pi')| \leq u$ whenever π' is obtained from π by transposing two elements. Then if π is chosen randomly from S_n then

$$\mathbf{Pr}(f(\pi) - \mathbf{E}(f) \leq -t) \leq \exp\left\{-\frac{2t^2}{nu^2}\right\}. \tag{14}$$

For a proof see e.g., Section 3.2 of [19] or Lemma 11 of [12].

In this context, think of choosing a random m-subset of $[n]$ as choosing a random π and then taking the first m elements as your subset.

Plugging in the estimates on $\mathbf{E}(Z_a^*)$ and Δ_a stated above in (14), we get that **qs** for every $a \in A_i$,

$$\begin{aligned} Z_a^* &\geq (1-\epsilon)\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{2\ell-k}{\ell}\right)^{2\ell-k}\binom{n-2(k-\ell)}{2\ell-k}p - n^{2\ell-k-1/2}p\log n \\ &\geq \left(1 - \frac{3}{2}\epsilon\right)\left(\frac{2\ell-k}{\ell}\right)^{2\ell-k}\binom{n}{2\ell-k}p . \end{aligned} \tag{15}$$

So assume that Y_i is chosen so that (15) holds. Now we expose the random permutation τ_i of Y_i. Let Z_a be the degree of a in G_i, which is the number of edges $E \in \mathcal{E}$ such that

1. $E \cap Y_i = E - S$ (the number of such edges is Z_a^*);

2. $E \cap Y_i$ forms a block $Y_{i,b}$ under τ_i;

3. E is labeled by i (this happens independently and with probability $1/f(E) \geq 1/f_0$).

Hence,

$$\mathbf{E}(Z_a) \geq Z_a^* \frac{\nu_\ell}{\binom{N_Y}{2\ell-k}} \frac{1}{f_0} \,.$$

Observe that changing τ_i by a single transposition changes the value of Z_a by at most 2 (at most two blocks $Y_{i,b}$ are affected by such a change). Therefore, applying concentration results for permutation graphs we get that for any $t > 0$

$$\mathbf{Pr}(Z_a \leq \mathbf{E}(Z_a) - t) \leq \exp\left\{-\frac{t^2}{2N_Y}\right\} \,.$$

Thus **qs** for every partition i and for every $a \in A_i$, its degree is G_i is at least

$$(1 - \frac{3}{2}\epsilon) \left(\frac{2\ell - k}{\ell}\right)^{2\ell - k} \binom{n}{2\ell - k} p \frac{\nu_\ell}{\binom{N_Y}{2\ell-k}} \frac{1}{f_0} - n^{1/2} \log n \geq (1 - 2\epsilon)\nu_\ell p_0 \,,$$

due to our assumption on ϵ.

The argument for the degrees of the vertices of B_i is quite similar. Fix $b \in B_i$. The vertex b corresponds to the block $Y_{i,b}$ of τ_i. Condition on $Y_{i,b} = S$ for some $S \subset [n]$, $|S| = 2\ell - k$. We expose a random subset X_i first. Let Z_b^* be the number of edges $E \in \mathcal{E}$ such that $S \subset E$ and $E \cap X_i = E - S$. For each edge $E \in N_H(S)$

$$\mathbf{Pr}(E \cap X_i = E - S) = \frac{\binom{n-k}{N_X - 2(k-\ell)}}{\binom{n-(2\ell-k)}{N_X}} = \frac{(N_X)_{2k-2\ell}}{(n - (2\ell - k))_{2k-2\ell}} = \left(1 + O\left(\frac{1}{n}\right)\right) \left(\frac{k - \ell}{\ell}\right)^{2k-2\ell} \,.$$

Therefore by assumption P_a,

$$\mathbf{E}(Z_b^*) \geq (1 - \epsilon) \left(1 + O\left(\frac{1}{n}\right)\right) \left(\frac{k - \ell}{\ell}\right)^{2k-2\ell} \binom{n - 2\ell + k}{2(k - \ell)} p \,.$$

Since changing the fate of one vertex with respect to the choice of X_i changes the value of Z_b^* by at most

$$\Delta_b = \max_{S' \in \binom{[n]}{2\ell-k+1}} d_H(S') \,,$$

and the latter quantity is bounded by $(1+\epsilon)\binom{n}{2k-2\ell-1}p$ by assumption P_a, we get by (14) that for any $t > 0$

$$\mathbf{Pr}(Z_b^* \leq \mathbf{E}(Z_b^*) - t) \leq \exp\left\{-\frac{2t^2}{n\Delta_b^2}\right\} \,.$$

Plugging in the estimates on $\mathbf{E}(Z_b^*)$ and Δ_b stated above, we get that **qs** for every $b \in B_i$,

$$\begin{aligned} Z_b^* &\geq (1-\epsilon)\left(1+O\left(\frac{1}{n}\right)\right)\left(\frac{k-\ell}{\ell}\right)^{2k-2\ell}\binom{n-2\ell+k}{2(k-\ell)}p - n^{2k-2\ell-1/2}p\log n \\ &\geq (1-\frac{3}{2}\epsilon)\left(\frac{k-\ell}{\ell}\right)^{2k-2\ell}\binom{n}{2k-2\ell}p\ . \end{aligned} \tag{16}$$

So assume that X_i is chosen so that (16) holds. Now we expose the random permutation σ_i of X_i. Let Z_b be the degree of b in G_i, which is the number of edges $E \in \mathcal{E}$ such that

1. $E \cap X_i = E \setminus S$ (the number of such edges is Z_b^*);
2. $E \cap X_i$ forms two consecutive blocks $X_{i,a}$, $X_{i,a+1}$ under σ_i;
3. E is labeled by i (this happens independently and with probability $1/f(E) \geq 1/f_0$).

Hence,

$$\mathbf{E}(Z_b) \geq Z_b^* \frac{\nu_\ell}{\binom{N_X}{2k-2\ell}}\frac{1}{f_0}\ .$$

Observe that changing σ_i by a single transposition changes the value of Z_b by at most 4. Therefore, applying again concentration results for permutation graphs we get that for any $t > 0$

$$\mathbf{Pr}(Z_b \leq \mathbf{E}(Z_b) - t) \leq \exp\left\{-\frac{t^2}{8N_X}\right\}\ .$$

Thus **qs** for every partition i and for every $b \in B_i$, its degree is G_i is at least

$$(1-\frac{3}{2}\epsilon)\left(\frac{k-\ell}{\ell}\right)^{2k-2\ell}\binom{n}{2k-2\ell}p\,\frac{\nu_\ell}{\binom{N_X}{2k-2\ell}}\,\frac{1}{f_0} - n^{1/2}\log n \geq (1-2\epsilon)\nu_\ell p_0\,,$$

due to our assumption on ϵ.

Now we treat typical co-degrees in the graph G_i. First fix $b_1, b_2 \in B_i$ and Y_{i,b_1}, Y_{i,b_2} and expose a random set X_i. Let Z_{b_1,b_2}^* be the number of subsets $S_1 \subset [n]$ of cardinality $|S_1| = 2(k-\ell)$ such that $S_1 \subset X_i$ and both $S_1 \cup Y_{i,b_1}$ and $S_1 \cup Y_{i,b_2}$ form an edge in $\mathcal{E}$. By our assumption P_b,

$$\mathbf{E}(Z_{b_1,b_2}^*) \leq (1+\epsilon)\left(\frac{k-\ell}{\ell}\right)^{2k-2\ell}\binom{n-2(2\ell-k)}{2(k-\ell)}p^2\,.$$

Using assumption P_a we see that changing X_i by one element changes Z_{b_1,b_2}^* by at most

$$\Delta_{b_1,b_2} = \max_{S \in \binom{[n]}{2\ell-k+1}} |N_H(S)| \leq (1+\epsilon)\binom{n}{2(k-\ell)-1}p.$$

Applying (14) we see that **qs** for every $b_1, b_2 \in B_i$,

$$Z^*_{b_1,b_2} \leq \left(1 + \frac{3}{2}\epsilon\right) \left(\frac{k-\ell}{\ell}\right)^{2(k-\ell)} \binom{n}{2(k-\ell)} p^2 . \tag{17}$$

Assume X_i is chosen so that (17) holds. Expose the random permutation σ_i of X_i. Let Z_{b_1,b_2} be the co-degree of b_1, b_2 in G_i, which is the number of blocks $X_{i,a}$ of X_i under σ_i such that $E_1 = X_{i,a} \cup X_{i,a+1} \cup Y_{i,b_1}, E_2 = X_{i,a} \cup X_{i,a+1} \cup Y_{i,b_2} \in \mathcal{E}$, and both edges E_1, E_2 are labeled by i. Then, recalling that an edge $E \in \mathcal{E}$ is labeled by i with probability $\frac{1}{f(E)} \leq \frac{1}{\left(1-\frac{3}{2}\epsilon\right) f_0}$, we get

$$\mathbf{E}(Z_{b_1,b_2}) \leq Z^*_{b_1,b_2} \frac{\nu_\ell}{\binom{N_X}{2k-2\ell}} \frac{1}{\left(1-\frac{3}{2}\epsilon\right)^2 f_0^2} .$$

Transposing one pair of elements of σ_i changes Z_{b_1,b_2} by at most 4. Using (14) again, we obtain that **qs** for every partition i and every pair $b_1, b_2 \in B_i$, the co-degree Z_{b_1,b_2} of b_1, b_2 in G_i satisfies:

$$\begin{aligned} Z_{b_1,b_2} &\leq \left(1 + \frac{3}{2}\epsilon\right) \left(\frac{k-\ell}{\ell}\right)^{2k-2\ell} \binom{n}{2k-2\ell} p^2 \frac{\nu_\ell}{\binom{N_X}{2k-2\ell}} \frac{1}{\left(1-\frac{3}{2}\epsilon\right)^2 f_0^2} - n^{1/2}\log n \\ &\leq (1+5\epsilon)\nu_\ell p_0^2 . \end{aligned}$$

Now consider $a_1, a_2 \in A_i$ and $X_{i,a_1}, X_{i,a_1+1}, X_{i,b_2}, X_{i,b_2+1}$ and expose a random set Y_i. Let $Z^*_{a_1,a_2}$ be the number of subsets $S_1 \subset [n]$ of cardinality $|S_1| = 2\ell - k$ such that $S_1 \subset Y_i$ and both $S_1 \cup X_{i,a_1} \cup X_{i,a_1+1}$ and $S_1 \cup X_{i,a_2} \cup X_{i,a_2+1}$ form an edge in $\mathcal{E}$. By our assumption P_b,

$$\mathbf{E}(Z^*_{a_1,a_2}) \leq (1+\epsilon) \left(\frac{2\ell-k}{\ell}\right)^{2\ell-k} \binom{n}{2\ell-k} p^2 .$$

Using assumption P_a we see that changing Y_i by one element changes $Z^*_{a_1,a_2}$ by at most

$$\Delta_{a_1,a_2} = \max_{S \in \binom{[n]}{2(k-\ell)+1}} |N_H(S)| \leq (1+\epsilon)\binom{n}{2\ell-k-1} p.$$

Applying (14) we see that **qs** for every $b_1, b_2 \in B_i$,

$$Z^*_{a_1,a_2} \leq \left(1 + \frac{3}{2}\epsilon\right) \left(\frac{k-\ell}{\ell}\right)^{2k-2\ell} \binom{n}{2k-2\ell} p^2 . \tag{18}$$

Assume Y_i is chosen so that (18) holds. Expose the random permutation τ_i of Y_i. Let Z_{a_1,a_2} be the co-degree of a_1, a_2 in G_i, which is the number of blocks $Y_{i,b}$ of Y_i under τ_i such that

$E_1 = X_{i,a_1} \cup X_{i,a_1+1} \cup Y_{i,b}, E_2 = X_{i,a_2} \cup X_{i,a_2+1} \cup Y_{i,b} \in \mathcal{E}$, and both edges E_1, E_2 are labeled by i. Then, recalling that an edge $E \in \mathcal{E}$ is labeled by i with probability $\frac{1}{f(E)} \leq \frac{1}{\left(1-\frac{3}{2}\epsilon\right) f_0}$, we get

$$\mathbf{E}(Z_{a_1,a_2}) \leq Z^*_{a_1,a_2} \frac{\nu_\ell}{\binom{N_Y}{2\ell-k}} \frac{1}{\left(1-\frac{3}{2}\epsilon\right)^2 f_0^2} \ .$$

Transposing one pair of elements of τ_i changes Z_{a_1,a_2} by at most 4. Using (14) again, we obtain that **qs** for every partition i and every pair $a_1, a_2 \in B_i$, the co-degree Z_{a_1,a_2} of a_1, a_2 in G_i satisfies:

$$\begin{aligned} Z_{a_1,a_2} &\leq \left(1+\frac{3}{2}\epsilon\right)\left(\frac{2\ell-k}{\ell}\right)^{2\ell-k}\binom{n}{2\ell-k}p^2\frac{\nu_\ell}{\binom{N_Y}{2\ell-k}}\frac{1}{\left(1-\frac{3}{2}\epsilon\right)^2 f_0^2} - n^{1/2}\log n \\ &\leq (1+5\epsilon)\nu_\ell p_0^2 \ . \end{aligned}$$

□

We can now apply Lemma 2 below with $n = \nu_\ell, d = p_0, \theta = 5\epsilon$ to show that each G_i contains at least $(1-(5\epsilon)^{1/3})\nu_\ell p_0 \geq (1-2\epsilon^{1/3})m/rn$ edge-disjoint perfect matchings. This completes the proof of Theorem 3 for the case $k/2 < \ell < k$.

6.2 $\ell = k/2$ (Theorem 4).

We first choose f_0 such that

$$\frac{\log n}{\epsilon^2} \ll f_0 \ll \epsilon^{3/4} p n^{1/8} \log^{1/8} n.$$

We then choose

$$r = \frac{(2-\epsilon)\binom{n}{k} f_0}{\nu_\ell^2}.$$

The random variable $f(E)$ is distributed as $Bin(r,\rho)$ where $\rho = \frac{\binom{\nu_\ell}{2}}{\binom{n}{k}}$ and so with the above parameter definitions, (9) holds **whp**.

Lemma 8 *Let G be a ν vertex, (p,ϵ)-regular graph. Suppose that G_0 is the random subgraph of G where each edge e of G is included independently with probability p_e, where $(1-\epsilon)p^* \leq p_e \leq p^*$. Suppose that*

$$\epsilon^2 \nu p p^* \gg \log \nu \text{ and } \epsilon^3 \nu p p^* \gg \log 1/\epsilon.$$

Then G_0 is $((1-2\epsilon)pp^*, 2\epsilon pp^*)$-regular, **qs**.

Proof The degree of vertex v in G_0 dominates $Bin(p\nu, (1-\epsilon)p^*)$ and so Property Q_a holds from Chernoff bounds. Similarly the number of edges between two sets S, T dominates $Bin((p-\epsilon)|S|\,|T|, (1-\epsilon)p^*)$ and is dominated by $Bin((p-\epsilon)|S|\,|T|, p^*)$ and Property Q_b also holds from Chernoff bounds. □

We apply Lemma 8 to each $G_{\mathcal{P}_i}$, where $\mathcal{P}_i$ is the partition induced by σ_i. Here $p^* = (1-\epsilon)^{-1} f_0$. Now by assumption $G_{\mathcal{P}_i}$ is (p, ϵ) regular. It follows that **qs**, G_i is $((1-3\epsilon)p/f_0, 2\epsilon p/f_0)$ regular. Applying Theorem 1 and using the upper constraint on f_0 we see that **qs** each G_i contains $(p(1-3\epsilon)/2f_0 - 8\epsilon p/f_0)\nu_\ell$ edge disjoint type ℓ Hamilton cycles. Thus H contains r times as many edge disjoint Hamilton cycles, verifying this case.

6.3 $\ell = k$ (Theorem 5).

Here the aim is to find many edge-disjoint perfect matchings. We first choose f_0 such that (12) holds and then choose

$$r = (1-\epsilon)\binom{n}{k}\frac{f_0}{\nu_k^2}.$$

We claim (see Lemmas 2 and 9) that **whp** each G_i will contain at least

$$n_0 = (1 - 2\epsilon^{1/3})\nu_k p_0$$

edge-disjoint perfect matchings. So H will contain at least rn_0 edge-disjoint perfect matchings, completing the proof of Theorem 5.

Lemma 9 **Whp**, *over our random choices of* $X_i, Y_i, \sigma_i, \tau_i$, *each* G_i *has minimum degree at least* $(1-2\epsilon)\nu_k p_0$ *and maximum co-degree at most* $(1+5\epsilon)\nu_k p_0^2$.

Proof The arguments here are very similar to those in Lemma 7, so we will be rather brief. We fix i and focus on G_i. We first show that the minimum degree in G_i is large. For $a \in A_i$, denote by Z_a its degree in G_i. Then, using assumption P_a and martingale-type arguments, we can show that

$$\mathbf{E}(Z_a) \geq \left(1 - \frac{3}{2}\epsilon\right)\left(\frac{k_Y}{k}\right)^{k_Y}\binom{n}{k_Y} p \frac{\nu_k}{\binom{N_Y}{k_Y}} \frac{1}{f_0}\,.$$

Using concentration results for permutation graphs again, we derive that **qs** for every partition i and every $a \in A_i$, the degree of a in G_i is at least

$$\left(1 - \frac{3}{2}\epsilon\right)\left(\frac{k_Y}{k}\right)^{k_Y}\binom{n}{k_Y} p \frac{\nu_k}{\binom{N_Y}{k_Y}} \frac{1}{f_0} - n^{1/2}\log n \geq (1-2\epsilon)\nu_k p_0\,,$$

due to our assumption on ϵ.

Let now Z_b denote the degree of vertex $b \in B_i$ in G_i. We can argue similarly, while invoking assumption P_a to show that **qs** for every partition i and every $b \in B_i$,

$$Z_b \geq (1-2\epsilon)\nu_k p_0 \,.$$

Finally, we treat the co-degrees of pairs of vertices in G_i. Let $b_1, b_2 \in B_i$. Let Z_{b_1,b_2} be their co-degree in G_i. Then using assumption R_e and martingale-type concentration arguments, we can prove that **qs** for every partition i and every pair of vertices $b_1, b_2 \in B_i$

$$\begin{aligned} Z_{b_1,b_2} &\leq (1+\frac{3}{2}\epsilon)\left(\frac{k_X}{k}\right)^{k_X}\binom{n}{k_X}p^2\frac{\nu_k}{\binom{N_X}{k_X}}\frac{1}{\left(1-\frac{3}{2}\epsilon\right)^2 f_0^2} - n^{1/2}\log n \\ &\leq (1+5\epsilon)\nu_k p_0^2 \,. \end{aligned}$$

Similarly, if $a_1, a_2 \in A_i$, let Z_{a_1,a_2} be their co-degree in G_i. Then using assumption P_b and martingale-type concentration arguments, we can prove that **qs** for every partition i and every pair of vertices $a_1, a_2 \in A_i$

$$\begin{aligned} Z_{a_1,a_2} &\leq (1+\frac{3}{2}\epsilon)\left(\frac{k_Y}{k}\right)^{k_Y}\binom{n}{k_Y}p^2\frac{\nu_k}{\binom{N_Y}{k_Y}}\frac{1}{\left(1-\frac{3}{2}\epsilon\right)^2 f_0^2} - n^{1/2}\log n \\ &\leq (1+5\epsilon)\nu_k p_0^2 \,. \end{aligned}$$

□

7 Concluding remarks

In this paper we have derived sufficient conditions for packing almost edges of k-uniform random and pseudo-random hypergraphs into disjoint type ℓ Hamilton cycles. This appears to be a first result of this kind for the problem of packing Hamilton cycles in this setting. There is no reason to believe our assumptions on the edge probability $p(n)$ or the density of a pseudo-random hypergraph are tight, and it would be quite natural to try and extend them and to obtain tight(er) bounds.

In the paper [11] we were able to show how to use the results of [9] in a game theoretic setting. More precisely, we showed how to play a Maker-Breaker type of game on the complete graph where Maker is able to construct an $(1/2-\epsilon, \epsilon)$-regular graph, $\epsilon = o(1)$. We could then use the results of [9] to show that Maker could construct approximately $n/4$ edge-disjoint Hamilton cycles when alternately choosing edges against an adversary. The techniques of that paper can be extended to the hypergraph setting in a straightforward manner.

Acknowledgement. The authors wish to thank the anonymous referees for their helpful criticism and remarks.

References

[1] R. F. Bailey and B. Stevens, *Hamiltonian decompostions of complete k-uniform hypergraphs*, Discrete Mathematics 310 (2010), 3088–3095.

[2] D. Bal and A.M. Frieze, Packing tight Hamilton cycles in uniform hypergraphs, to appear.

[3] Zs. Baranyai, On the factorization of the complete uniform hypergraph, In: *Infinite and finite sets*, Vol. I, Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, 91–108.

[4] B. Bollobás, **Random graphs**, 2nd ed., Vol. 73, Cambridge University Press, Cambridge, 2001.

[5] F. R. K. Chung, R. L. Graham, and R. M. Wilson, *Quasi-random graphs*, Combinatorica 9 (1989), 345–362.

[6] A. Dudek and A.M. Frieze, *Loose Hamilton cycles in random k-uniform hypergraphs*, Electronic Journal of Combinatorics, to appear.

[7] A. Dudek and A.M. Frieze, *Tight Hamilton cycles in random uniform hypergraphs*, submitted.

[8] A.M. Frieze, Loose Hamilton cycles in random 3-uniform hypergraphs, *Electronic Journal of Combinatorics* 17, N28.

[9] A.M. Frieze and M. Krivelevich, *On packing Hamilton cycles in ϵ-regular graphs*, Journal of Combinatorial Theory Ser. B 94 (2005) 159-172.

[10] A.M. Frieze, M. Krivelevich and P. Loh, *Packing tight Hamilton cycles in 3-uniform hypergraphs*, Proceedings of the 22nd Symposium on Discrete Algorithms (SODA'11), 913–932.

[11] A.M. Frieze, M. Krivelevich, O. Pikhurko and T. Szabo, *The game of JumbleG*, Combinatorics, Probability and Computing 14 (2005) 783-794.

[12] A.M. Frieze and B. Pittel, Perfect matchings in random graphs with prescribed minimal degree, *Trends in Mathematics, Birkhauser Verlag, Basel* (2004) 95-132.

[13] H. Hàn and M. Schacht, *Dirac-type results for loose Hamilton cycles in uniform hypergraphs*, Journal of Combinatorial Theory Ser. B 100 (2010), 332-346.

[14] S. Janson, T. Łuczak and A. Ruciński, **Random graphs**, Wiley, New York, 2000.

[15] A. Johansson, J. Kahn and V. Vu, *Factors in random graphs*, Random Structures and Algorithms 33 (2008), 1–28.

[16] P. Keevash, D. Kühn, R. Mycroft and D. Osthus, *Loose Hamilton cycles in hypergraphs*, manuscript.

[17] M. Krivelevich and B. Sudakov, Pseudo-random graphs, In: *More sets, graphs and numbers*, E. Györi, G. O. H. Katona, L. Lovász, Eds., Bolyai Soc. Math. Studies Vol. 15, 199–262.

[18] D. Kühn, R. Mycroft and D. Osthus, *Hamilton ℓ-cycles in uniform hypergraphs*, Journal of Combinatorial Theory Ser. A 117 (2010), 910-927.

[19] C. McDiarmid, Concentration, in: *Probabilistic Methods for Algorithmic Discrete Mathematics*, Algorithms Combin., 16, Springer, Berlin, 1998, 195–248.

[20] V. Rödl, A. Ruciński and E. Szemerédi, *An approximate Dirac-type theorem for k-uniform hypergraphs*, Combinatorica 28 (2008) 229 - 260.

[21] A. Thomason, Pseudorandom graphs. In: *Random graphs '85 (Poznań, 1985)*, North-Holland Math. Stud., 144, North-Holland, Amsterdam, 1987, 307–331.

A Claimed improvement for Theorem 1

The proof in [9] requires

1. $\epsilon^2 n/\alpha \gg \log n$ and $\epsilon^3 n/\alpha \gg \log(1/\epsilon)$ for the proof of Lemma 1.
2. $\alpha\epsilon^3 \gg \frac{1}{(n \log n)^{1/2}}$ is needed to verify condition Q2.

With these inequalities, the proof of Theorem 1 of [9] goes through.