

Sequentially constrained Hamilton Cycles in random graphs

Alan Frieze* and Wesley Pegden†
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh PA 15213

Abstract

We discuss the existence of Hamilton cycles in the random graph $G_{n,p}$ where there are restrictions caused by (i) coloring sequences, (ii) a subset of vertices must occur in a specific order and (iii) there is a bound on the number of inversions in the associated permutation.

1 Randomly colored random graphs

In this paper we consider several questions related to Hamilton cycles in random graphs. Our first set of questions arise from randomly coloring the edges or vertices. Suppose we are given a graph $G = (V, E)$, k colors $1, 2, \dots, k = O(1)$ and a map $c : E \rightarrow [k]$. A color pattern will be a sequence $\mathbf{c} = (c_1, c_2, \dots, c_n)$. Our first result concerns edge colored copies of $G_{n,p}$. Given a sequence \mathbf{c} we say that the Hamilton cycle $H = (x_1, x_2, \dots, x_n, x_1)$ (as a sequence of vertices) is \mathbf{c} -colored if $c(\{x_i, x_{i+1}\}) = c_i$ for $i = 1, 2, \dots, n$.

Suppose that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ where $\alpha_1, \dots, \alpha_k$ are constants and $\alpha_1 + \dots + \alpha_k = 1$ and $\alpha_i > 0, i = 1, 2, \dots, k$. Let $\beta = \min\{\alpha_i : i \in [k]\}^{-1}$ and let $G_{n,p;\boldsymbol{\alpha}}$ denote the random graph $G_{n,p}$ where each edge is independently given a random color i from the palette $[k]$ with probability α_i .

Theorem 1. *Let \mathbf{c} be an arbitrary sequence of colors. Let $p = (\log n + \omega)/n$ where $\omega \rightarrow \infty$. Then w.h.p. $G_{n,\beta p;\boldsymbol{\alpha}}$ contains a \mathbf{c} -colored Hamilton cycle.*

Remark 1. *In the above theorem we are allowed to take $c_i = \ell, i = 1, 2, \dots, n$ for each possible $\ell \in [k]$ and so we cannot improve the βp probability threshold. This is because each subgraph induced by a single color must itself be Hamiltonian.*

Remark 2. *As will be seen, the proof of Theorem 1 can be repeated verbatim for the random digraphs $D_{n,p}$ and $D_{n,\beta p;\boldsymbol{\alpha}}$.*

Remark 3. *The proof can also be extended without difficulty to deal with Hamilton cycles in edge colored hypergraphs. Here we must let p be the threshold probability for a particular type of Hamilton cycle. These thresholds are known fairly precisely for all except loose Hamilton cycles. See Frieze [13], Dudek, Frieze, Loh and Speiss [7] for loose Hamilton cycle thresholds and Dudek and Frieze [6] and Narayanan and Schacht [18] for the remaining types.*

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One can also consider problems where the vertices are what carry the colors. Here our results are less tight. Suppose now that there are k colors and each $v \in [n]$ is given a color $c(v) \in [k]$. Let $V_i = \{v : c(v) = i\}$ and assume that $|V_i| = \alpha_i n$ for $i \in [k]$ where α is as in Theorem 1 so that each set V_j is of linear size. We denote this randomly colored graph by $G_{n,p}^{[k]}$. We can assume w.l.o.g. that vertices $1, 2, \dots, \alpha_1 n$ are given color 1 and vertices $\alpha_1 n + 1, \alpha_1 n + 2, \dots, (\alpha_1 + \alpha_2)n$ are given color 2 etc. Given a sequence \mathbf{c} we now say that the Hamilton cycle $H = (x_1, x_2, \dots, x_n, x_1)$ (as a sequence of vertices) is \mathbf{c} -colored if $c(x_i) = c_i$ for $i = 1, 2, \dots, n$.

Theorem 2. *Let \mathbf{c} be an arbitrary sequence of colors where each color j appears exactly $\alpha_j n$ times. Let $p = K \log n/n$ where $K = K(k)$ is sufficiently large. Then w.h.p. $G_{n,p}^{[k]}$ contains a \mathbf{c} -colored Hamilton cycle.*

We can expand our results by coloring the edges as well as the vertices. We prove two results along these lines. First, suppose that $q \geq n$ and we randomly color each edge with one of q colors. A Hamilton cycle is rainbow colored if each edge has a different color. Using a result of Bell, Frieze and Marbach [2] we can strengthen Theorem 2 to

Theorem 3. *Let \mathbf{c} be an arbitrary sequence of colors where each color j appears exactly $\alpha_j n$ times. Let $p = K \log n/n$ where $K = K(k)$ is sufficiently large. Suppose in addition that the edges of $G_{n,p}$ are randomly colored with one of $q \geq n$ colors. Then w.h.p. $G_{n,p}^{[k]}$ contains a \mathbf{c} -colored rainbow Hamilton cycle.*

We can also consider a combination of Theorems 1 and 2.

Theorem 4. *Let $\mathbf{c}_1 = (c_{1,1}, c_{1,2}, \dots, c_{1,n})$ be an arbitrary sequence of colors from the palette $[k]$ and let $\mathbf{c}_2 = (c_{2,1}, c_{2,2}, \dots, c_{2,n})$ be another arbitrary sequence of colors from the palette $[\ell]$ where each color $j \in [\ell]$ appears exactly $\beta_j n$ times. Let $p = K \log n/n$ where $K = K(k, \ell)$ is sufficiently large. Suppose that each edge of $G_{n,p}$ is given a random color from palette $[k]$, using distribution α and exactly $\beta_j n = \Omega(n)$ vertices are given color j for $j \in [\ell]$. Denote this coloring of $G_{n,p}$ by $G_{n,p;\alpha}^{[\ell]}$. Then w.h.p. $G_{n,p;\alpha}^{[\ell]}$ contains a Hamilton cycle in which the edges follow pattern \mathbf{c}_1 and the vertices follow pattern \mathbf{c}_2 .*

1.1 Prior work on randomly colored random graphs

Rainbow Hamilton Cycles The most well-studied case is that of rainbow Hamilton cycles. Here we are given $k \geq n$ colors which are applied randomly to the edges of $G_{n,p}$. A rainbow Hamilton cycle is one where each edge has a different color. Cooper and Frieze [5] showed that if $k \geq 20n$ and $p \geq \frac{20 \log n}{n}$ then a randomly colored $G_{n,p}$ contains a rainbow hamilton cycle w.h.p. This was improved to $k \geq n + o(n)$ and $p \sim \frac{\log n}{n}$ by Frieze and Loh. Currently the strongest result is that of Ferber and Krivelevich [10] who prove a hitting time result when $k = n + o(n)$.

Repeating Patterns Special cases of Theorem 1 were proved by Espig, Frieze and Krivelevich [9] and by Anastos and Frieze [1]. Here the sequence \mathbf{c} is required to consist of the repetition of some fixed bounded length subsequence. In this case it was possible to prove hitting time results. Chakraborti, Frieze and Hasabnis [4] proved a hitting time version for the existence of patterns where the Hamilton cycle is required to decompose into k concatenated mono-chromatic paths.

2 A fixed order for a subset of vertices

Here we consider the following problem. We have a fixed set $S_0 \subseteq [n]$ and a fixed ordering of the vertices in S_0 and we wish to determine the likelihood that there is a Hamilton cycle that goes through S_0 in the given order. We do not require that the vertices of S_0 be visited consecutively. Without loss of generality we can assume that $S_0 = [s_0]$ and that we wish to find S_0 in the natural order.

Theorem 5. *Let $p = (\log n + \log \log n + \omega)/n$, $\omega = o(\log \log n)$ and $s_0 = \omega n / \log n$ where $\omega = o(\log \log \log n)$. Then w.h.p. $G_{n,p}$ contains a Hamilton cycle in which the vertices S_0 appear in natural order.*

2.1 Prior work

The closest result to this is the result of Robinson and Wormald [19]. They consider random regular graphs and ask for Hamilton cycles that contain a prescribed number of $o(n^{2/5})$ edges must be contained in order in the cycle.

Our final result concerns Hamilton cycles where we place a restriction on the number of inversions in the permutation of $[n]$ that it defines. So we treat a Hamilton cycle H as a sequence $\sigma = (i_1 = 1, i_2, \dots, i_n)$ and we define $\iota(H) = |\{k < \ell : i_k > i_\ell\}|$.

Theorem 6. *Suppose that $M = \Omega(n \log n)$. There is a constant K such that if $p \geq \frac{Kn \log n}{M}$ then w.h.p. $G_{n,p}$ contains a Hamilton cycle H with $\iota(H) \leq M$. Furthermore, if $p \leq (1 - \varepsilon) \min\{\frac{\log n}{n}, \frac{n}{eM}\}$ then w.h.p. $G_{n,p}$ contains no such Hamilton cycle. Here ε is an arbitrary positive constant.*

We get a restricted rainbow version almost for free:

Theorem 7. *Suppose that the edges of $G_{n,p}$ are randomly colored with one of $q \geq n$ colors. There is a constant $K = K(\varepsilon)$ such that if $p \geq \frac{K \log n}{n}$ then w.h.p. $G_{n,p}$ contains a rainbow Hamilton cycle H with $\iota(H) \leq \varepsilon n^2$.*

In general, except for the case $M = \Omega(n^2)$, there is a $\log n$ gap between the upper and lower bound in Theorem 6. (The gap is smaller for $M = \Omega(n^2)/\omega$, $\omega = o(\log n)$.) We will be able to remove this gap by studying a greedy algorithm from Frieze and Pegden [15].

Theorem 8. *If $M \leq Kn^2 / \log^2 n$ and $p \geq \frac{10 \max\{K, 1\}n}{M}$ then w.h.p. $G_{n,p}$ contains a Hamilton cycle H with $\iota(H) \leq M$.*

3 Proof of Theorem 1

Let $N = \binom{n}{2}$ and consider the following sequence of (partially) edge colored graphs Γ_m , $m = 0, 1, \dots, N$. Let e_1, e_2, \dots, e_N be an enumeration of the edges of K_n . Each edge $e_i = \{x_i, y_i\}$ gives rise to k colored copies $e_i^{(\ell)}$, $\ell \in [k]$. To construct Γ_t we include e_1, e_2, \dots, e_t independently with probability kp and give each included edge a random color using distribution α . Then for $i > t$ we include each edge independently with probability p . Thus Γ_0 is a copy of $G_{n,p}$ and Γ_N is a copy of $G_{n, kp; \alpha}$.

A Hamilton cycle $H = (e_{\pi(i)}, i = 1, 2, \dots, n)$ (as a sequence of edges) of Γ_t is (\mathbf{c}, t) -proper if $c(e_{\pi(j)}) = c_j$ for $\pi(j) \leq t$. Let \mathcal{G}_t denote the set of graphs containing a (\mathbf{c}, t) -proper Hamilton cycle.

Lemma 9.

$$\mathbb{P}(\Gamma_t \in \mathcal{G}_t) \leq \mathbb{P}(\Gamma_{t+1} \in \mathcal{G}_{t+1}) \text{ for } t \geq 0.$$

Proof. We use a modification of the coupling argument of McDiarmid [17]. The status of edge e_i consists of (i) whether or not it is included and (ii) its color if $i \leq t$. We condition on the *identical* status of the edges $e_i, i \neq t+1$ in Γ_t, Γ_{t+1} and argue about the conditional probability of both graphs having a (\mathbf{c}, t) -proper Hamilton cycle. Denote these conditional probabilities by p_t, p_{t+1} respectively. The conditional probability space is now just the status of e_{t+1} in Γ_t, Γ_{t+1} . We argue that $p_t \leq p_{t+1}$. Let $\widehat{\Gamma}$ denote the subgraph induced by the edges $e_i, i \neq t+1$ whose status means they are included in Γ_t and Γ_{t+1} . (Thus $\widehat{\Gamma}$ is only partially edge colored.) There are several cases:

1. $\widehat{\Gamma} \in \mathcal{G}_t \cap \mathcal{G}_{t+1}$. In this case $p_t = p_{t+1} = 1$.
2. $\widehat{\Gamma} + e_{t+1} \notin \mathcal{G}_t \cup \mathcal{G}_{t+1}$, regardless of the status of e_{t+1} . In this case $p_t = p_{t+1} = 0$.
3. Failing 1. and 2. we consider the case where $\widehat{\Gamma}$ is such that the existence of the edge e_{t+1} matters. We consider the event (i) that including e_{t+1} creates a (\mathbf{c}, t) -proper Hamilton cycle $P + e_{t+1}$ in Γ_t and the event (ii) that including e_{t+1} with an appropriate color creates a (\mathbf{c}, t) -proper Hamilton cycle $P + e_{t+1}$ in Γ_{t+1} . In this case we see that

$$p_{t+1} = \mathbb{P}((ii)) \geq \min \{\beta p \alpha_i : i \in [k]\} \geq p = \mathbb{P}((i)) = p_t.$$

□

This proves Theorem 1.

4 Proof of Theorems 2 and 3

For this theorem we will use the breakthrough result of Frankston, Kahn, Narayanan and Park [12]. Recall the setup in [12]: A hypergraph \mathcal{H} (thought of as a set of edges) is r -bounded if $e \in \mathcal{H}$ implies that $|e| \leq r$. For a set $S \subseteq X = V(\mathcal{H})$ we let $\langle S \rangle = \{T : S \subseteq T \subseteq X\}$ denote the subsets of X that contain S . Let $\langle \mathcal{H} \rangle = \bigcup_{H \in \mathcal{H}} \langle H \rangle$ be the collection of subsets of X that contain an edge of \mathcal{H} . We say that \mathcal{H} is κ -spread if we have the following bound on the number of edges of \mathcal{H} that contain a particular set S :

$$|\mathcal{H} \cap \langle S \rangle| \leq \frac{|\mathcal{H}|}{\kappa^{|S|}}, \quad \forall S \subseteq X. \tag{1}$$

Let X_p denote a subset of X where each $x \in X$ is included independently in X_p with probability p . The following theorem is from [12]:

Theorem 10. *Let \mathcal{H} be an r -bounded, κ -spread hypergraph and let $X = V(\mathcal{H})$. There is an absolute constant $C > 0$ such that if*

$$p \geq \frac{C \log r}{\kappa} \tag{2}$$

then w.h.p. X_p contains an edge of \mathcal{H} . Here w.h.p. assumes that $r \rightarrow \infty$.

Bell, Frieze and Marbach [2] proved a rainbow version of Theorem 10.

Theorem 11. *Let \mathcal{H} be an r -bounded, κ -spread hypergraph and let $X = V(\mathcal{H})$ be randomly colored from $Q = [q]$ where $q \geq r$. Suppose also that $\kappa = \Omega(r)$. There is an absolute constant $C > 0$ such that if*

$$p \geq \frac{C \log r}{\kappa} \tag{3}$$

then w.h.p. X_p contains a rainbow colored edge of \mathcal{H} . Here w.h.p. assumes that $r \rightarrow \infty$ (and thus $\kappa \rightarrow \infty$).

(In truth the general theorem in [2] only proves that for a given $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such if $C \geq C_\varepsilon$ then X_p contains a rainbow colored edge of \mathcal{H} with probability $1 - \varepsilon$. We have to apply a Theorem of Friedgut [16] (second remark following Theorem 2.1 of that paper) to obtain w.h.p.)

In our use of Theorem 11 we let $X = \binom{[n]}{2}$. Each $x = \{u, v\} \in X$ will have colored endpoints $\{c(u), c(v)\}$. Our hypergraph \mathcal{H} consists of sets of n edges with colored endpoints that together make up a \mathbf{c} -colored Hamilton cycle. We now check that (2) holds with $\kappa = \Omega(n)$.

We first observe that $|\mathcal{H}| = \frac{1}{h} \prod_{j=1}^k n_j!$ where $n_i = |V_i|$ and h is the number of automorphisms of a \mathbf{c} -colored Hamilton cycle. Fix a set S for which $\phi(S) := |\mathcal{H} \cap \langle S \rangle| > 0$. In particular, S is the edge-set of a collection of paths. We bound $\phi(S)$ as follows. If S consists of the edges of t paths, then to choose a Hamilton cycle consistent with S , we must

- Choose an orientation of each of the t paths;
- Choose, for the starting vertex of each of the t paths, where its index in $1, \dots, n$ in the Hamilton cycle, whose color must match the color of the starting vertex;
- Choose, for each vertex not incident with any edge in S , its index in $1, \dots, n$ in the Hamilton cycle, whose color must match the color of the starting vertex.

Of course many such choices will not give rise to valid \mathbf{c} -colored Hamilton cycles, but any valid \mathbf{c} -colored Hamilton cycles consistent with S can be specified by such choices.

Note that, as above, after choosing the orientation of the paths, we have to choose the index of at most $n - s$ vertices where $s = |S|$, since the number of components (paths or isolated vertices) of the graph induced by the set S is $n - s$, and we only choose the index of the first vertex in each path. In particular, after choosing the orientation of the path, suppose we need to choose the index of $n_j - s_j$ vertices of color j for each j , where $\sum s_j = s$. Now, as $n_i = \alpha_i n, j = 1, \dots, k$, we have

$$\frac{1}{h} \prod_{j=1}^k (n_j - s_j)! = |\mathcal{H}| \prod_{j=1}^k \frac{(n_j - s_j)!}{n_j!} \leq |\mathcal{H}| \prod_{j=1}^k \frac{e^{s_j^2/n_j}}{n_j^{s_j}} \leq e^s |\mathcal{H}| \prod_{j=1}^k \frac{1}{n_j^{s_j}} \leq e^s |\mathcal{H}| \frac{1}{n^s \alpha_{\min}^s}.$$

This gives that

$$\phi(S) \leq 2^s |\mathcal{H}| \frac{e^s}{n^s \alpha_{\min}^s}$$

So, (2) holds with $r = n$ and $\kappa = \alpha_{\min} n / 2e$. This proves Theorem 3 (which implies Theorem 2, perhaps with a smaller hidden constant C).

5 Proof of Theorem 4

For the proof of this theorem we combine McDiarmid's coupling with Theorem 2. We use the notation of Section 3. We define the sequence $\Gamma_1, \Gamma_2, \dots, \Gamma_N$ similarly to how we did in that section but with the difference that we have colored the vertices as claimed.

A Hamilton cycle $H = (e_{\pi(i)}, i = 1, 2, \dots, n)$ (as a sequence of edges) of Γ_t and equal to $(v_i, i = 1, 2, \dots, n)$ (as a sequence of vertices) is $(\mathbf{c}_1, \mathbf{c}_2, t)$ -proper if $c(e_{\pi(j)}) = c_{1,j}$ for $\pi(j) \leq t$ and $c(v_i) = c_{2,i}$ for $i = 1, 2, \dots, n$. Let \mathcal{G}_t denote the set of graphs containing a $(\mathbf{c}_1, \mathbf{c}_2, t)$ -proper Hamilton cycle.

Lemma 12.

$$p_t = \mathbb{P}(\Gamma_t \in \mathcal{G}_t) \leq p_{t+1} = \mathbb{P}(\Gamma_{t+1} \in \mathcal{G}_{t+1}) \text{ for } t \geq 0.$$

Proof. The proof of this is identical to that of the proof of Lemma 9 except that we replace (\mathbf{c}, t) -proper by $(\mathbf{c}_1, \mathbf{c}_2, t)$ -proper. \square

Now Γ_0 is distributed as $G_{n,p}^{[\ell]}$ and Theorem 2 states that a $(\mathbf{c}_1, \mathbf{c}_2, 0)$ -proper Hamilton cycle exists w.h.p. On the other hand, Γ_N is distributed as $G_{n,p,\alpha}^{[\ell]}$ and the lemma implies that it contains a $(\mathbf{c}_1, \mathbf{c}_2, N)$ -proper Hamilton cycle which is what we need to prove. This proves Theorem 4.

6 Proof of Theorem 5

We begin by generating $G = \bigcup_{i=1}^4 \Gamma_i$ where each $\Gamma_i, i = 1, 2, 3, 4$ is an independent copy of G_{n,p_i} . Here $p_3 = p_4 = \omega/4n$ and $p_1 = p_2$ and $1 - p = \prod_{i=1}^4 (1 - p_i)$. It follows that $p_1 = p_2 \sim p/2$. For $v \in [n]$, we let $d(v)$ denote the degree of vertex v in $G_{n,p}$ and $d_i(v), i = 1, 2$ denote the degree of v in Γ_i .

Let $G_1 = \Gamma_1 \cup \Gamma_2$. G_1 has minimum degree at least 2, w.h.p. We let $\text{SMALL} = \{v : d_{G_1}(v) \leq \frac{1}{40} \log n\}$. An easy first moment calculation implies that w.h.p.

S(i) $|\text{SMALL}| \leq n^{1/3}$.

S(ii) $v, w \in \text{SMALL}$ implies that $\text{dist}(v, w) \geq 5$.

S(iii) No cycle of length less than 5 contains a vertex of SMALL.

The calculations supporting this claim can be found in Lemma 3.1 of [3]. We note next that w.h.p. $G_{n,p}$ has maximum degree at most $5 \log n$. This follows from a simple first moment calculation, again given in Lemma 3.1 of [3].

Similarly, we let $\text{TINY}_0 = \{v : d_1(v) \leq \frac{1}{40} \log n\}$ and initialise $\text{TINY} = \text{TINY}_0$. We remove vertices from Γ_1 and place them in TINY when their Γ_1 -degrees become smaller than $\frac{1}{80} \log n$. We will show that TINY_0 is small in Lemma 13 below. Initialise the set $\text{AVOID} = \text{SMALL} \cup N(\text{SMALL}) \cup \text{TINY}$.

The construction of our Hamilton cycle goes as follows.

Step 1 For $v = 1, 2, \dots, s_0$ we construct a set of vertex disjoint paths $P_v = (x_v, \dots, v, \dots, y_v)$ where $x_v, y_v \notin \text{TINY}$. (If $v \notin \text{TINY}$ then we can simply let $x_v = y_v = v$ and $P_v = v$). These paths are of length at most 6. These paths will avoid using vertices in $\text{AVOID} \cup S_0 \cup \bigcup_{w < v} V(P_w)$. Also, for $v \geq 2$, we avoid using vertices in $N(x_1)$. All edges except perhaps those incident with x_v, y_v will be from Γ_2 . After we create a path, we delete the vertices in the interior of the path and their incident edges.

Step 2 We then use the edges of Γ_1 to construct vertex disjoint paths Q_v from y_v to x_{v+1} for $v = 1, 2, \dots, s_0 - 1$. They will be of length at most $4 \log n / \log \log \log n$. These paths will avoid using vertices in $\text{AVOID} \cup S_0$. After we create a path, we delete the vertices in the interior of the path and their incident edges. If after this deletion the Γ_1 -degree of a vertex becomes at most $\log n / 80$ then we add it to TINY and update AVOID .

Step 3 We let $P^* = (P_1, Q_1, P_2, Q_2, \dots, Q_{s_0-1}, P_{s_0})$ and let $x^*, y^* \notin \text{TINY}$ be its endpoints. Here $x^* = x_1$ and $y^* = y_{s_0}$.

Step 4 We then use the extension-rotation algorithm to find a Hamilton cycle that contains P^* as a subpath.

We note that because of our bound on s_0 ,

$$\sum_{v=1}^{s_0} (|V(P_v)| + |V(Q_v)|) \leq s_0 \left(7 + \frac{4 \log n}{\log \log \log n} \right) = o(n). \quad (4)$$

6.1 Analysis of Step 1

We first show that $|\text{TINY}_0|$ is small.

Lemma 13. $|\text{TINY}_0| \leq n^{2/3}$ *w.h.p.*

Proof. We have

$$\begin{aligned} \mathbb{E}(|\text{TINY}_0|) &\leq n \sum_{k=0}^{\log n/40} \binom{n}{k} p_1^k (1-p_1)^{n-1-k} \leq n \sum_{k=0}^{\log n/40} \left(\frac{e^{1+o(1)} \log n}{2k} \right)^k n^{-1/2} \\ &\leq 2n(20e^{1+o(1)})^{\log n/40} n^{-1/2} \leq n^{-1/3}. \end{aligned}$$

The lemma now follows from the Markov inequality. \square

Fix $v \in S_0$. Expose the edges of G_1 incident with v and let A_1 denote the other endpoints of these edges. Assume first that $v \in \text{TINY} \setminus \text{SMALL}$, so that $|A_1| \geq \frac{1}{40} \log n$. We go through the vertices of A_1 in order until we find a vertex a_v with a Γ_2 -neighbor $x_v \notin \text{AVOID} \cup N(x_1) \cup \bigcup_{j=1}^{v-1} V(P_j)$. Lemma 13 and Lemma 15 below ensures that $|\text{TINY}| \leq 2n/3$, which, together with A, B, and C, imply that

$$|\text{AVOID} \cup N(x_1) \cup \bigcup_{j=1}^{v-1} V(P_j)| \leq 2n/3 + o(n).$$

If η_v denotes the number of trials to find a_v then η_v is dominated by a geometric random variable with success probability at least $(1 - (1 - p_2)^{n/3+o(n)})$ and so $\mathbb{P}(\eta_v \geq 10) = o(n^{-2})$. This verifies the existence of a_v, x_v . The rest of P_v is justified similarly. We just find another path, avoiding a_v, b_v, x_v as well. If $v \in \text{TINY} \cap \text{SMALL}$ then we choose two arbitrary neighbors of v , which will not be in SMALL by S(ii) and grow paths to $[n] \setminus \text{TINY}$, avoiding $\text{AVOID} \cup N(x_1) \cup \bigcup_{j=1}^{i-1} V(P_j)$.

6.2 Analysis of Step 2

We first remove vertices in AVOID from Γ_1 . We construct Q_1, Q_2, \dots, Q_{s_0} in this order and at each step $v \geq 1$, we do the following in the graph Γ_1 : we remove the vertices of Q_{v-1} (if $v \geq 2$) and then repeatedly remove vertices of degree (in Γ_1) at most $\frac{1}{80} \log n$ until what remains has minimum degree at least $\frac{1}{80} \log n$. (Removed vertices are placed into TINY.) We show that w.h.p.

Property 1 $|V(\Gamma_1)| = \Omega(n)$ throughout.

Property 2 The diameter of Γ_1 is $O(\log n / \log \log \log n)$ throughout.

We begin with the following lemma:

Lemma 14. *Suppose that $S \subseteq [n]$. In the graph G ,*

$$|S| \leq \sigma_0 = \frac{5n \log \log n}{\log n} \text{ implies that } e(S) \leq \frac{7|S| \log n}{\log \log n}. \quad (5)$$

$$|S| \geq \sigma_0 \text{ implies that } e(S) \leq \frac{3|S|^2 \log n}{2n}. \quad (6)$$

Proof. Let $s = |S|$. Then,

$$\begin{aligned} \mathbb{P}(\exists S : e(S) \geq \alpha s) &\leq \binom{n}{s} \binom{s^2/2}{\alpha s} p^{\alpha s} \\ &\leq \left(\frac{ne}{s} \cdot \left(\frac{se^{1+o(1)} \log n}{2\alpha n} \right)^{\alpha s} \right)^s. \end{aligned}$$

For (5) we use $\alpha = 7 \log n / \log \log n$ and for (6) we use $\alpha = 3s \log n / 2n$. □

We now argue that Property 1 holds throughout.

Lemma 15. *W.h.p. we have $|V(\Gamma_1)| \geq 0.39n$ throughout.*

Proof. Suppose we delete γn vertices belonging to paths and then repeatedly remove vertices of degree at most $\frac{1}{80} \log n$ from Γ_1 . Here γn bounds the total length of all the paths Q_1, Q_2, \dots, Q_{s_0} and is $o(n)$, see (4). Initially, Γ_1 has at least $(0.49 - o(1))n \log n / 2$ edges and after κn small-degree-vertex removals Γ_1 has $(1 - \gamma - \kappa)n$ vertices and at least $(0.245 - o(1) - 5\gamma - \kappa/80)n \log n$ edges. (We lose at most $5 \log n$ edges per path vertex and at most $\log n / 80$ edges per low degree vertex.) It follows from (6) that w.h.p.

$$\left(0.245 - o(1) - \frac{\kappa}{80}\right) \leq \frac{3(1 - \gamma - \kappa)^2}{2} \text{ implying that } \left(0.245 - o(1) - \frac{1}{80}\right) \leq \frac{3(1 - \gamma - \kappa)^2}{2}.$$

It follows from this that w.h.p. Γ_1 still has at least $(1 - \gamma - \kappa)n \geq 0.39n$ vertices. □

This verifies Property 1.

Lemma 16. *W.h.p., the diameter of Γ_1 is at most $4 \log n / \log \log \log n$ throughout.*

Proof. Fix $x, y \in V(\Gamma_1)$ and let S_i denote the set of vertices at distance i from x in Γ_1 and define T_i similarly for y . Fix i and let $S = S_i \cup S_{i+1}$. Suppose that $|S| \leq \frac{5n \log \log n}{\log n}$. Then because the minimum degree in S is at least $\log n/80$,

$$\frac{|S_i| \log n}{160} \leq e(S) \leq \frac{7|S| \log n}{\log \log n}.$$

It follows that $|S_{i+1}| \geq \frac{1}{2000}|S_i| \log \log n$. Let $i_0 \leq 3 \log n / 2 \log \log \log n$ be the smallest positive integer such that $(\frac{1}{2000} \log \log n)^{i_0} \geq 5n \log \log n / \log n$. If $S_{i_0} \cap T_{i_0} \neq \emptyset$ then there is a path of length at most $4 \log n / \log \log \log n$ from x to y in Γ_1 . On the other hand, in Γ_1 ,

$$\begin{aligned} & \mathbb{P} \left(\exists S, T; |S|, |T| \geq \frac{5n \log \log n}{\log n}, e(S : T) = S \cap T = \emptyset \right) \\ & \leq \binom{n}{5n \log \log n / \log n}^2 (1 - p_1)^{(5n \log \log n / \log n)^2} \\ & \leq \left(\frac{e \log n}{5 \log \log n} \right)^{10n \log \log n / \log n} \exp \left\{ -\frac{12(\log \log n)^2 n}{\log n} \right\} = o(1). \end{aligned}$$

It follows that the diameter of Γ_1 is $4 \log n / \log \log \log n$. (We have used the fact that Γ_1 is always an induced subgraph of the initial Γ_1 .) \square

This verifies Property 2.

Now we argue that we can construct the paths Q_i . Given Properties 1 and 2, we use the edges of Γ_2 to find a short paths from x_i and y_i to what's left of $V(\Gamma_1)$ at the time of the construction of Q_i . The path Q_i will be decomposed into $Q_{i,j}, j = 1, 2, 3$ where $Q_{i,1}$ uses Γ_2 -edges and is from x_i to $u_i \in V(\Gamma_1)$, $Q_{i,2}$ is from u_i to $v_i \in V(\Gamma_1)$ in Γ_1 and $Q_{i,3}$ uses Γ_2 -edges and is from v_i to y_i . We let A_j denote the set of vertices at distance exactly j from x_i using paths that avoid using $\bigcup_{j=1}^{i-1} V(Q_j)$. Because $x_i \notin \text{TINY}$, we have $|A_1| \geq \log n/40$ and then given A_j , $|A_{j+1}|$ dominates the binomial $\text{Bin}(n - |\text{AVOID}| - |A_1| - \dots - |A_{j-1}| - o(n), 1 - (1 - p_2)^{|A_j|})$. So w.h.p. $|A_2| = \Omega(\log^2 n)$ and $A_3 \cap V(\Gamma_1) \neq \emptyset$, using Lemma 15. This verifies the existence of $Q_{i,1}$ and $Q_{i,3}$ is dealt with similarly.

In summary Step 2 constructs a path P^* of length $O(s_0 \log n / \log \log \log n) = o(n)$.

6.3 Analysis of Step 3

We only need to verify that $x^* = x_1, y^* = y_{s_0} \notin \text{TINY}$. $x_1^* \notin \text{TINY}$ because it was not in TINY_0 and we avoid using vertices in $N(x_1)$. y_{s_0} is selected to be not in TINY at the end of the process.

6.4 Analysis of Step 4

Let G_1^* be obtained from G_1 after contracting P^* to an edge $e^* = \{x^*, y^*\}$ and deleting any edge $\{u, v\}$ that is incident with an interior vertex of P^* , but is not an edge of P^* . We let $V_1^* = V(G_1^*)$. We note that the minimum degree in G_1^* is at least 2. We then let $G_3^* = G_1^* \cup \Gamma_3$.

We consider the usual extension-rotation algorithm for finding a Hamilton cycle. We apply it to G_3^* and we use Γ_4 as *boosters*. We begin with a longest path in $P_0 = (x_1, x_2, \dots, x_k)$ in G_3^* that contains e^* and we

consider *restricted* rotations that do not delete e^* . Given a path $P = (x_1, x_2, \dots, x_k)$ and an edge $\{x_k, x_i\}$ where $1 < i < k - 1$ we say that the path $Q = (x_1, \dots, x_{i-1}, x_i, x_k, x_{k-1}, \dots, x_{i+1})$ is obtained from P by a restricted rotation if $e^* \neq \{x_i, x_{i+1}\}$. x_1 is called the fixed endpoint.

Suppose then that END is the set of vertices that occur as endpoints of paths obtainable from P_0 by a sequence of restricted rotations. Since P_0 was a longest path containing e^* , the neighbors of END in G_3^* are all vertices of P_0 . We show that we have something close to the usual Posá property. For $S \subseteq V_1^*$ we let $N^*(S)$ denote the neighbors of S in G_1^* .

Lemma 17. $|N^*(\text{END})| \leq 2|\text{END}| + 1$.

Proof. We show that

$$|N^*(\text{END}) \setminus \{x^*, y^*\}| \leq 2|\text{END}| - 1, \quad (7)$$

This implies the lemma. If $u \in \text{END}$ and $v \in P_0 \setminus (\text{END} \cup \{x^*, y^*\})$ is a neighbor of u in G_1 then one of v 's neighbors in P_0 must be in END. This is because when the rotations produce a path P_u with u as an endpoint, another rotation will make one of v 's P_v neighbors an endpoint. If both of these neighbors are P_0 neighbors then we are done. Otherwise the rotations have deleted a P_0 -edge $\{x, v\} \neq e^*$ containing v . This means x or v is in END. Our hypothesis excludes $v \in \text{END}$. This completes the proof of (7). \square

To apply the usual arguments, we prove

Lemma 18. *The following hold w.h.p.:*

- (a) $S \subseteq V_1^*$, $|S| \leq n/1000$ implies that $|N^*(S)| \geq \sum_{v \in S_1} d(v) + 2|S_2|$, where $S_1 = S \cap \text{SMALL}$ and $S_2 = S \setminus S_1$.
- (b) G_3^* is connected.

Proof. (a) Suppose first that $S \cap \text{SMALL} = \emptyset$ and $|N^*(S)| \leq 5|S|$. Let $T = S \cup N^*(S)$. Vertices in S have degree at least $\log n/40$ in G_3^* and so $e(T) \geq |S| \log n/80 \geq |T| \log n/480$. If $|T| \leq \frac{5n \log \log n}{\log n}$ then this contradicts (5). Otherwise, (6) is contradicted, unless $3|T|^2 \log n/2n \geq |T| \log n/480$ which implies that $|S| \geq n/1000$.

Suppose now that $S \subseteq V_1^*$ with $|S| \leq n/1000$. Then from the properties S(ii), S(iii) claimed at the beginning of the proof of Theorem 5, we have

$$\begin{aligned} |N^*(S)| &\geq |N^*(S_1)| + |N^*(S_2)| - |N^*(S_1) \cap S_2| - |N^*(S_2) \cap S_1| - |N^*(S_1) \cap N^*(S_2)| \\ &\geq \sum_{v \in S_1} d(v) + 5|S_2| - |S_2| - |S_2| - |S_2| \\ &= \sum_{v \in S_1} d(v) + 2|S_2|. \end{aligned} \quad (8)$$

(b) We first claim that w.h.p., the graph Γ_3 consists of a giant component plus $o(n)$ small components of size at most $n_0 = ne^{-\omega/10}$. To verify the claim, let X_k denote the number of components in Γ_3 of size $k \in [n_0, n/2]$. Then

$$\mathbb{E} \left(\sum_{k=n_0}^{n/2} X_k \right) \leq \sum_{k=n_0}^{n/2} \binom{n}{k} k^{k-2} \left(\frac{\omega}{4n} \right)^{k-1} \left(1 - \frac{\omega}{4n} \right)^{k(n-k)} \leq n \sum_{k=n_0}^{n/2} \left(\frac{e^{1-\omega/8} k \omega}{n_0} \right)^k \leq n^2 (e^{-\omega/50})^{n_0} = o(1).$$

So, w.h.p. there are no components of size in the range $[n_0, n/2]$. We also have

$$\mathbb{E} \left(\sum_{k=1}^{n_0} kX_k \right) \leq n \sum_{k=1}^{n_0} k \left(\frac{e^{1-\omega/8}\omega}{k} \right)^k = O(ne^{-\omega/10}).$$

The Markov inequality implies that there are $o(n)$ vertices in components of size at most n_0 . So w.h.p. Γ_3 has a unique giant component of size $n - o(n)$.

Part (a) shows that the minimum component size in G_3^* is at least $n/1000$. This combined with the fact that Γ_3 consists of a giant component of size $n - o(n)$ proves Part (b). \square

Now because $d_{G_3^*}(v) \geq 2$ for $v \in [n]$, we can see from Lemma 17 and (8) that $|\text{END}| > n/1000$ unless $\text{END} \subseteq \text{SMALL}$ and at most one vertex of END has degree more than 2. We rule out this possibility. Suppose that $v \in \text{END} \cap \text{SMALL}$. Then v is not adjacent to x^* or y^* , by construction. One more rotation will bring a vertex of $[n] \setminus \text{SMALL}$ into END , contradiction.

It follows from Lemmas 17 and 18(b) that w.h.p. $|\text{END}| \geq cn, c = 1/1000$. For each $v \in \text{END}$, we can define a set $\text{END}(v)$ of at least cn vertices obtainable by doing rotations with v as the fixed endpoint.

We can now use a standard argument, see for example Chapter 6.2 of [11], to use Γ_4 to create the required Hamilton cycle. It will be convenient to replace the edges of Γ_4 by $\mu = \omega n/10$ random edges $\{f_1, f_2, \dots, f_\mu\}$. These edges are independent of G_3^* . Starting with $i = 0$ we construct a sequence of paths P_0, P_1, \dots, P_s where s is a Hamilton path and construct a Hamilton cycle from there. Given P_i we do restricted rotations until either (i) we construct a path P , one of whose endpoints has a neighbor outside P or (ii) we construct at least cn sets $\text{END}(v)$, each of size at least cn . In the former case (i) we just extend P to a path P_{i+1} which has one more vertex than P . In the latter case (ii) we go to the next edge f_j in the sequence f_1, f_2, \dots, f_μ to see if it is of the form $\{x, y\}, x \in \text{END}(y)$. This closes a path to a cycle. The probability of this is at least c^2 . Given such a cycle C and the fact that G_3^* is connected, there are two possibilities: (a) C is a Hamilton cycle or (b) there is an edge $\{x, y\}$ such that $x \in V(C)$ and $y \notin V(C)$. We can delete an edge $e \neq e^*$ of C such that we obtain a new path with endpoint y that is one edge longer than P_i . This will be our P_{i+1} . The probability this process fails is at most the probability that μ trials with success probability c^2 fails to produce n successes, which is $e^{-\Omega(n)}$.

This completes the proof of Theorem 5.

7 Proof of Theorem 6

We will use Theorem 10 to prove this. We must first prove bounds on the number of Hamilton cycles H with a bound on $\iota(H)$. Denote this upper bound on $\iota(H)$ by M . For a sequence $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ we let $\mu_k = |\{j < k : \sigma_j > \sigma_k\}|$ for $k = 1, 2, \dots, n$. Then $0 \leq \mu_j < j$ for each j , and $\iota(\sigma) = \sum_{j=1}^n \mu_j$. In particular, we have that

$$|\{\sigma : \iota(\sigma) \leq M\}| = \left| \left\{ (\mu_1, \dots, \mu_{n-1}) : \sum_{j=0}^{n-1} \mu_j \leq M, 0 \leq \mu_j < j \right\} \right|; \quad (9)$$

indeed, there is a bijection between the sets on the left and the right, realized by building a permutation iteratively in the order $1, 2, \dots, n$ and placing k so that it occurs in front of μ_k previously allocated elements.

We are seeking an upper bound on the threshold probability for the existence of a particular type of Hamilton cycle and so it is acceptable to restrict our attention to a more restrictive subclass of Hamilton cycles. So we restrict our attention to those cycles for which

$$\sum_{j=1}^n \mu_j \leq M \text{ and } 0 \leq \mu_j < \begin{cases} j & j \leq M/n. \\ \frac{M}{n} & j > M/n. \end{cases} \quad (10)$$

To apply Theorem 10 we let \mathcal{H} denote the set of Hamilton cycles H such that (10) holds. Thus,

$$|\mathcal{H}| = \left(\frac{M}{n}\right)! \left(\frac{M}{n}\right)^{n-M/n}. \quad (11)$$

We prove below that for $S \subseteq X$, $X = \binom{[n]}{2}$, $|S| = s$,

$$|\langle S \rangle| \leq \left(\frac{M}{n}\right)! \left(\frac{M}{n}\right)^{n-M/n-s}. \quad (12)$$

It follows from (11) and (12) that

$$\frac{|\mathcal{H}|}{|\langle S \rangle|} \geq \left(\frac{M}{en}\right)^s.$$

The upper bound on the existence threshold in Theorem 6 now follows from Theorem 10 with $r = \binom{n}{2}$ and $\kappa = M/n$. To obtain the upper bound in Theorem 7 we apply Theorem 11 in place of Theorem 10.

Proof of (12): As in the proof of Theorem 4 the set S defines a collection of vertex disjoint sub-paths P_1, P_2, \dots, P_k of any Hamilton cycle that contains S . Given such a path P_i we let x_i denote the lowest numbered vertex of P_i . We see that once we have chosen μ_{x_1} , the remaining values $\mu_{x_i}, i \geq 2$ are *constrained* by the edges of the cycle H that are not on P . Let V_0 denote the set of first vertices of P_1, P_2, \dots, P_k and let $S_1 = \bigcup_{i=1}^k V(P_i) \setminus V_0$ and $S_2 = S_2 \cap [M/n]$. Then,

$$|\langle S \rangle| \leq \left(\frac{M}{n}\right)^{n-M/n-(|S_1|-|S_2|)} \left(\frac{M}{n} - |S_2|\right)! \quad (13)$$

$$\leq \left(\frac{M}{n}\right)^{n-M/n-\max\{0, s-M/n\}} \left(\frac{M}{n} - \min\left\{s, \frac{M}{n}\right\}\right)!. \quad (14)$$

The second factor in (13) follows from the additional fact that given the μ -values of the elements of S_2 there will be $|S_2|$ values forbidden as a μ -value for the unconstrained elements of $[M/n]$. These forbidden values are those that would insert the element into the interior of a path. Equation (12) follows from (14).

8 Proof of Theorem 8

We first write $G_{n,p} = G_1 \cup G_2$ where the G_i are independent copies of G_{n,p_i} , where $p_1 = p/3$ and $1-p = (1-p_1)(1-p_2)$. Note that $p_2 \sim 2p/3$. We begin by constructing a path P_0 via the following algorithm: We start with $v_1 = 1$. Then for $j \geq 1$ we let

$$\phi(j) = \min \{k \in N : k \notin \{v_1, v_2, \dots, v_j\} \text{ and } \{v_j, k\} \in E(G_1)\}$$

and let $v_{j+1} = \phi(j)$ i.e. we move from v_j to the lowest index k that has not been previously added to P_0 . We define U_j by

$$U_j = \{i \leq n : i \notin \{v_1, v_2, \dots, v_j\}\}$$

We stop the process at $j = j_0$ when either $|U_j| = \frac{2 \log n}{p_1}$ or v_{j+1} does not exist. We then extend the path $P_0 = (v_1, v_2, \dots, v_{j_0})$ to a Hamilton cycle H using the edges of G_2 to create a path through U .

Observe first that if $|U| = k > \frac{2 \log n}{p_1}$ then $\mathbb{P}(j_0 \leq n - k) \leq n(1 - p_1)^k$. This is because at j_0 we find that v_{j_0} has no neighbors in the set of unvisited vertices U and the existence of such edges is unconditioned at this point. Thus w.h.p.

$$|U| = \frac{2 \log n}{p_1} \text{ and } j_0 = n - \frac{2 \log n}{p_1}. \quad (15)$$

Next let $j_1 = \min \{j : j \in U\}$. Then $\mathbb{P}(j_1 \leq k) \leq n \mathbb{E}_{j_0} (1 - p_1)^{j_0 - k}$. This is because $j_1 \leq k$ implies that $j_0 - k$ non-edges have been reported for vertex j_1 . So, w.h.p.,

$$j_1 \geq n - \frac{2 \log n}{p_1}. \quad (16)$$

Now let $\alpha_j = |\{k > j : v_k < v_j\}|$ for all $1 \leq j \leq n$, so that $\iota(H) = \alpha_1 + \alpha_2 + \dots + \alpha_n$. If we can complete $(v_1, v_2, \dots, v_{j_0})$ to a Hamilton cycle H , then

$$\iota(H) \leq \alpha_1 + \alpha_2 + \dots + \alpha_{j_0} + |U|(n - j_1).$$

Next we define an approximation a_j to α_j . We let $a_j = |\{t < v_j : t \notin V_j\}|$ for all $j \geq 1$, where $V_j = \{v_1, v_2, \dots, v_j\}$. Observe that $\alpha_j \leq a_j$ for $j \leq j_0$. Moreover,

$$\mathbb{P}(a_j = k) = (1 - p_1)^k p_1 \text{ for } k \geq 0. \quad (17)$$

To see this, observe that the vertex v_j was chosen as the leftmost vertex available to the algorithm at round j , and determining this vertex involves querying edges which have not yet been conditioned by the running of the algorithm. Observe that (17) holds even when conditioning on any previous history of the algorithm.

So $a_1 = 0$ and a_2, a_3, \dots is a sequence of independent copies of $Geo(p_1) - 1$ where $Geo(p_1)$ is the geometric random variable with probability of success p_1 . We thus have:

$$\mathbb{E} \left(\sum_{j=0}^{j_0} \alpha_j \right) \leq \mathbb{E} \left(\sum_{j=0}^{j_0} a_j \right) \leq \mathbb{E} \left(\sum_{j=0}^n a_j \right) \leq n \frac{1 - p_1}{p_1}. \quad (18)$$

Moreover, standard concentration arguments give that $\sum_{j=0}^{j_0} \alpha_j \leq 2n/p_1$ w.h.p. So, if we can complete $(v_1, v_2, \dots, v_{j_0})$ to a Hamilton cycle H , then w.h.p.

$$\iota(H) \leq \frac{2n}{p_1} + \frac{4 \log^2 n}{p_1^2} \leq M, \quad (19)$$

given that

$$p \geq \frac{10 \max \{K, 1\} n}{M}.$$

All that remains it to show that using the edges of G_2 , we can w.h.p. extend $(v_1, v_2, \dots, v_{j_0})$ to a Hamilton cycle. For this, we only have to show that there is a Hamilton path in the sub-graph Γ of G_2 induced by U that can be added to P_0 to create a Hamilton cycle through $[n]$.

Let $N = \frac{2 \log n}{p_1}$ and observe that $p_2 \geq \frac{4 \log N}{3N}$. Indeed,

$$\frac{4 \log N}{3N p_2} = \frac{4(\log 2 + \log \log n + \log 1/p_1)p_1}{3p_2 \log n} \lesssim \frac{2}{3}.$$

It follows from standard results (see Chapter 6 of [11]) that there is a positive constant $c > 0$ such that w.h.p. there are in Γ , cN vertices x_1, x_2, \dots, x_{cN} such that for each i there are cN Hamilton paths with one endpoint x_i and otherwise distinct endpoints. So the probability we cannot add a Hamilton path in Γ to P_0 is at most $2(1 - p_2)^{cN} = o(1)$. This completes the proof of Theorem 8.

9 Comments and open problems

While Theorems 1 – 3 are fairly general they can be improved in at least two ways. First we can ask for hitting time versions where we wait for sufficiently many edges and colors. Second and more challenging would be to prove that our random graphs *simultaneously* contain all possible sequences, rather than a specific one.

In the case of Theorem 5 the bound $s_0 = o(\log \log \log n)$ should probably be replaced by $s_0 = o(\log \log n)$ in line with the fact that most pairs of vertices in $G_{n,p}$, $p \sim \log n/n$ are $O(\log n / \log \log n)$ apart.

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