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SHORT COMMUNICATION

A COST FUNCTION PROPERTY FOR PLANT LOCATION PROBLEMS

A.M. FRIEZE

Queen Mary College, University of London, London, England

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1. A cost function property

We consider a general problem of finding a subset I of $M = \{1, ..., m\}$ which minimises a 'cost' function $C(I)$ which satisfies the following property.

Property P. *If $I \subseteq J$ and $s \notin J$, then*

$$C(I \cup \{s\}) - C(I) \leqslant C(J \cup \{s\}) - C(J) . \tag{1}$$

Property P is possessed by the simple plant location problem and more general problems as pointed out by Babayev [1], where $C(I)$ is the minimum 'delivery cost' plus 'construction cost' when the plants in set I are considered to be open.

For simplicity we shall replace $I \cup \{s\}$ by $I + s$ where no confusion is possible.

From Property P we may develop all the results on 'gain' functions used to solve simple plant location problems as well as the existence of a class of sub-optimal subsets. The proofs are straightforward and are omitted.

Theorem 1. *Let $S = \{s_1, ..., s_p\}$ be such that $S \cap I = \emptyset$, then*

$$C(I + s_1 + ... + s_p) - C(I) \geqslant \sum_{i=1}^{p} [C(I + s_i) - C(I)] . \tag{2}$$

Corollary 1. *If* $C(I+s) \geqslant C(I)$, $\forall s \notin I$, *then*

$$C(J) \geqslant C(I) \qquad \forall J \supseteq I\,. \tag{3}$$

Definition 1. We shall call any set having the property expressed in the above corollary, an OP1 set.

Theorem 2. *Let* $S = \{s_1, ..., s_p\}$ *be such that* $S \subseteq I$, *then*

$$C(I-S) - C(I) \geqslant \sum_{i=1}^{p} [C(I-s_i) - C(I)]\,. \tag{4}$$

Corollary 2. *If* $C(I-s) \geqslant C(I)$, $\forall s \in I$, *then*

$$C(J) \geqslant C(I) \qquad \forall J \subseteq I\,. \tag{5}$$

Definition 2. We shall call any set having the property expressed in the above corollary an OP2 set.

Clearly an optimal solution is both an OP1 set and an OP2 set.

Theorem 3. (a) *If* I *is an* OP1 *set and* $J \supseteq I$, *then* J *is also an* OP1 *set.* (b) *If* I *is an* OP2 *set and* $J \subseteq I$, *then* J *is also an* OP2 *set.*

The next theorem which generalises Theorems 1 and 2 is applicable in a generalised origin search [2].

Theorem 4. *Let the sets* I, S, T *be such that* $I \supseteq T$, $I \cap S = \emptyset$ *and let* $S = \{s_1, ..., s_p\}$, $T = \{t_1, ..., t_q\}$, *then*

$$C(I+S-T) - C(I) \geqslant \sum_{i=1}^{p} \{C(I-T+s_i) - C(I-T)\} + \sum_{j=1}^{q} \{C(I-t_j) - C(I)\}\,, \tag{6}$$

$$C(I+S-T) - C(I) \geqslant \sum_{i=1}^{p} \{C(I+s_i) - C(I)\} + \sum_{j=1}^{q} \{C(I+S-t_j) - C(I+S)\}\,. \tag{7}$$

2. Application to tree search algorithms

The use of gain functions in solving these problems is well known [2].

The properties of OP1 and OP2 sets can be used to curtail the search in the following way. If at a particular point ν in the search, Ω_ν is the set of plants fixed open, and F_ν is the set of free plants, then all sets I considered in forward steps from ν satisfy

$$\Omega_\nu \subseteq I \subseteq \Omega_\nu + F_\nu . \tag{8}$$

If I is an optimal solution, then clearly Ω_ν is OP2 and $\Omega_\nu + F_\nu$ is OP1. The search can thus be constrained to maintain the above properties.

3. Some results

We have tested the use of OP1 and OP2 sets in the simple plant location problem. We programmed two branch and bound algorithms both of which started with all plants closed.

A1. Gain function tests only.

A2. A1 + check $\Omega_\nu + F_\nu$ for OP1 after backtracking.

(This was found to be the best way of using Theorem 3 in our algorithm.) See for the results Table 1.

In general, it was found that the OP1 tests were quite effective if the delivery costs were no smaller than the plant costs and marginally inefficient otherwise.

The programs were written in FORTRAN and tested on a CDC 6600.

Table 1

Number of plants	Number of customers	Average plant cost	Average delivery cost	A1 time	A2 time
40	40	60	25	38	48
50	50	60	25	511	542
40	40	60	50	140	47
50	50	60	50	–	450
40	40	35	50	750	68
40	50	35	50	–	123

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References

[1] Dj.A. Babayev, "Comments on the note of Frieze", *Mathematical Programming* 7 (1974) 249–252.

[2] K. Spielberg, "Plant location with generalised search origin", *Operations Research* 18 (1970) 165–178.