Department of Mathematical Sciences
Carnegie Mellon University

21-366 Random Graphs
Test 1: Solutions

Name:____________________________________

Problem	Points	Score
1	40	
2	30	
3	30	
Total	100	

Q1: (40pts)
Suppose that $p = d/n$ where d is constant. Prove that w.h.p., in $G_{n,p}$, no vertex belongs to more than one triangle.

Solution: If there is a vertex v that lies in more than one triangle, then there is a set of $k = 4, 5$ vertices that contain at least $k + 1$ edges. The probability of this can be bounded by

$$\sum_{k=4}^{5} \binom{n}{k} 2^k \left(\frac{d}{n}\right)^{k+1} \leq \frac{1}{n} \sum_{k=4}^{5} \frac{2^k d^{k+1}}{k!} = o(1).$$

Q2: (30pts)
Suppose that $c \neq 1$ is constant and that $\epsilon_n = 1/\log\log n$. Show that w.h.p. the length of the longest path component in $G_{n,p}$, $p = \frac{c}{n}$ is $(1 \pm \epsilon_n)\frac{\log n}{c - \log c}$.
Solution: Let $L_\pm = (1 \pm \epsilon_n)\frac{\log n}{c-\log c}$ and let X_- denote the number of components that are paths of length L_- and let X_+ denote the number of components that are paths of length at least L_+.
Now we know that w.h.p. there are no path components of length more than $A \log n$ for some constant $A > 0$. This is because (a) there are no components of size greater than $A \log n$ and (b) the giant component has too many edges to be a path. Thus,

$$\begin{aligned}
\mathbf{P}(X_+ > 0) &\leq o(1) + \sum_{k=L_++1}^{A\log n} \binom{n}{k} k! \left(\frac{c}{n}\right)^{k-1} \left(1 - \frac{c}{n}\right)^{k(n-k)} \\
&\leq o(1) + \frac{(1+o(1))n}{c} \sum_{k=L_++1}^{A\log n} (ce^{-c})^k \\
&\leq o(1) + \frac{2An\log n}{c} \cdot (ce^{-c})^{L_+} \\
&\leq o(1) + \frac{2An\log n}{c} \cdot \frac{(ce^{-c})^{\epsilon_n L_+}}{n} \\
&= o(1).
\end{aligned}$$

Now, with $X = X_-$ and $L = L_-$,

$$\begin{aligned}
\mathbf{E}(X) &= \binom{n}{L+1}(L+1)! \left(\frac{c}{n}\right)^L \left(1 - \frac{c}{n}\right)^{L(n-L)-\binom{L+1}{2}+L} \\
&\geq (1 - o(1))n(ce^{-c})^L \\
&= (1 - o(1))(ce^{-c})^{-\epsilon_n L} \\
&\to \infty.
\end{aligned}$$

Furthermore, if $P_1, P_2, \ldots, P_M$ is an enumeration of the paths of length L in K_n and X_i is thge indicator for P_i being a path component then

$$\begin{aligned}
\mathbf{E}(X^2) &= \mathbf{E}(X) + \sum_{i \neq j} \mathbf{P}(X_i = X_j = 1) \\
&\leq \mathbf{E}(X) + \mathbf{E}(X)^2.
\end{aligned}$$

This is because if $i \neq j$ then

$$\mathbf{P}(X_i = X_j = 1) = \begin{cases} \mathbf{P}(X_i = 1)\mathbf{P}(X_j = 1) & P_i \cap P_j = \emptyset. \\ 0 & P_i \cap P_j \neq \emptyset. \end{cases}$$

Thus

$$\mathbf{P}X > 0 \geq \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)} \geq \frac{1}{\frac{1}{\mathbf{E}(X)} + 1} \to 1.$$

Q3: (30pts)
Let $\mathcal{C}$ denote the set of connected unicyclic graphs on vertex set $[n]$. Suppose that Z is the length of the unique cycle C_H in a randomly chosen member $H \in \mathcal{C}$. Show that, where $N = \binom{n}{2}$,

$$\mathbf{E}Z = \frac{n^{n-2}(N-n+1)}{|\mathcal{C}|}.$$

Hints: Count the number X of pairs (H, e) in two ways, where $e \in C_H$ and $H \in \mathcal{C}$. Let X_k denote the number of $H \in \mathcal{C}$ with $|C_H| = k$.
Solution: First,

$$X = \sum_{spanning\ trees\ T} |\{e \notin E(T)\}| = n^{n-2}(N-n+1).$$

This is because there are n^{n-2} spanning trees and each $e \notin E(T)$ creates a cycle when added to T.
On the other hand,

$$X = \sum_{k=1}^{n} kX_k$$

since each H with $|C_H| = k$ gives rise to k pairs, by deleting an edge of the unique cycle. Thus,

$$|\mathcal{C}|\mathbf{E}(Z) = |\mathcal{C}| \sum_{k=1}^{n} k\frac{X_k}{|\mathcal{C}|} = n^{n-2}(N-n+1).$$