Homework 1: Solutions

1.4.1 Suppose that $p = \frac{1}{n^{2/3}\omega}$ where $\omega \to \infty$. Let X denote the number of copies K_4 in $G_{n,p}$. Then

$$\mathbf{E}(X) = \binom{n}{4} p^6 \leq n^4 \cdot \frac{1}{n^4\omega^6} = \frac{1}{\omega^6} \to 0.$$

1.4.3 Let X denote the number of sets $S, 5 \leq |S| \leq n/\log n$ such that $e(S) \geq 2|S|$. Then

$$\begin{aligned}
\mathbf{E}(X) &\leq \sum_{s=5}^{n/\log n} \binom{n}{s}\binom{\binom{s}{2}}{2s} p^{2s} \\
&\leq \sum_{s=5}^{n/\log n} \left(\frac{ne}{s}\right)^s \left(\frac{s^2 e}{2s}\right)^{2s} \left(\frac{d}{n}\right)^{2s} \\
&= \sum_{s=5}^{n/\log n} \left(\frac{e^3 d^2 s}{4n}\right)^s \\
&\leq \sum_{s=5}^{n^{1/2}} \left(\frac{e^3 d^2}{4n^{1/2}}\right)^s + \sum_{s=n^{1/2}}^{n/\log n} \left(\frac{e^3 d^2}{4\log n}\right)^s \\
&= o(1).
\end{aligned}$$

1.4.6 Let $a = \lceil (\log n)^{1/2} \rceil$. Let X denote the number fo vertices of degree a. Then

$$\begin{aligned}
\mathbf{E}(X) &= n\binom{n-1}{a} p^a (1-p)^{n-1-a} \\
&\geq (1-o(1))n \left(\frac{np}{a}\right)^a e^{-np}.
\end{aligned}$$

For this we have used Lemma 21.1(f) and

$$(1-p)^{n-1-a} = \exp\{-(n-1-a)(p+O(p^2))\}.$$

Continuing, because $a^a = n^{o(1)}$, we have

$$\mathbf{E}(X) \geq n^{1-o(1)}.$$

For the second moment,

$$\begin{aligned}
\mathbf{E}(X^2) &= \mathbf{E}(X) + \mathbf{E}(X)\sum_{j=2}^{n} \mathbf{P}(deg(j) = a \mid deg(1) = a) \\
&= \mathbf{E}(X) + \mathbf{E}(X)\sum_{j=2}^{n} \mathbf{P}(deg(j) = a \text{ and } (1,j) \in E_{n,p} \mid deg(1) = a) + \\
&\qquad \mathbf{E}(X)\sum_{j=2}^{n} \mathbf{P}(deg(j) = a \text{ and } (1,j) \notin E_{n,p} \mid deg(1) = a) \\
&\le \mathbf{E}(X) + a\mathbf{E}(X) + \mathbf{E}(X)^2.
\end{aligned}$$

Here we use

$$\begin{aligned}
&\mathbf{P}(deg(j) = a \text{ and } (1,j) \notin E_{n,p} \mid deg(1) = a) \\
&\qquad = \binom{n-2}{a} p^a (1-p)^{n-1-a} \le \mathbf{P}(deg(j) = a).
\end{aligned}$$

So,

$$\mathbf{P}(X \ge 1) \ge \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)} \ge \frac{1}{\frac{1}{\mathbf{E}(X)} + \frac{a}{\mathbf{E}(X)} + 1} = 1 - o(1).$$