Notes on Combinatorial Optimization

August 25, 2019

1 Shortest path

1.1 Non-negative lengths

We are given a digraph $D = ([n], E)$ with vertex set $[n]$. Let $\mathcal{P}$ denote the set of paths in D and let $\ell : \mathcal{P} \to \mathbb{R}$. Think initially that there are edge lengths $\ell : E \to \mathbb{R}_+$ and that

$$\ell(P) = \ell_{reg}(P) = \sum_{e \in P} \ell(e).$$

Dijstra's Algorithm:
begin
for $i = 2, \ldots, n$, $d(i) \leftarrow \ell(1, i)$, $P_i \leftarrow (1, i); S_1 \leftarrow \{1\}$;
  for $k = 2, \ldots, n$ **do**;
  begin
    $d(i) = \min\{d(j) :\notin S_k\}$;
    $S_{k+1} \leftarrow S_k \cup \{i\}$;
    for $j \notin S_{k+1}$ **do**
    if $d(j) > \ell(P_i, j)$ then $d(j) \leftarrow \ell(P_i, j)$, $P_j \leftarrow (P_i, j)$;
  end
end

Lemma 1.1. *On termination of Dijstra's Algorithm, $d(i) = \ell(P_i)$ is the minimum length of a path from 1 to i, for all i*

Proof. At each stage we can verify by induction on k that for each $i \notin S_k$, $d(i)$ is the minimum length of a path from 1 to i for which all vertices but i are in S_k. If true for k then when we add vertex i we simply update the d's correctly.

Suppose that i is added at Step r. Let $P = (x_0 = 1, x_2, \ldots, x_m = i)$ be a path from 1 to i. Suppose that $x_0, x_1, \ldots, x_{l-1} \in S_{r-1}$ and $x_l \notin S_{r-1}$. Then,

$$\ell(P) \geq \ell(x_0, x_1, \ldots, x_l) \geq d(x_l) \geq d(i) = \ell(P_i).$$

□

Note now that all we have assumed about ℓ is that

$$P = (P_1, P_2) \text{ implies } \ell(P_1) \leq \ell(P). \quad (1)$$

In which case, we can apply the algorithm to solve problems where path length is defined as follows:

Time dependent path lengths: Suppose edge $e = (x, y)$ has two parameters $a_e, b_e \geq 0$ and that if we start a walk at time 0 and arrive at x at time t then the edge length is $a_e + b_e t$. Suppose that $P = (e_0, e_1, \ldots, e_k)$ as a sequence of edges and that $P_i = (e_0, e_1, \ldots, e_i)$. Then we now have $\ell(P_0) = a_{e_0}$ and $\ell(P_i) = a_{e_i} + b_{e_i} \ell(P_{i-1})$.

Visit S in a fixed order: S is a set of vertices and feasible paths must visit S in some fixed order. Individual edge lengths are non-negative. Then

$$\ell(P) = \begin{cases} \ell_{reg}(P) & P \cap S \text{ visited in correct order.} \\ \infty & Otherwise. \end{cases}$$

Avoid S: S is a set of vertices and there is a penalty of $f(k)$ for visiting S, k times. Here $f(k)$ is monotone increasing in k. Individual edge lengths are non-negative. Then $\ell(P) = \ell_{reg}(P) + f(|V(P) \cap S|$.

1.2 No negative cycles

Suppose first that for paths P is a path that begins at vertex 1 and x is an arbitrary vertex. Then we define

$$P * x = \begin{cases} (P, x) & x \notin P. \\ P(1, x) & x \in P. \end{cases}$$

Here $P(1, x)$ is the subpath of P from 1 to x.

Assumption: Suppose that P, Q are paths from vertex 1 to vertex y. Suppose that $x \notin P$ and that $\ell(Q) \leq \ell(P)$. Then $\ell(Q * x) \leq \ell(P, x)$.

Putting $P = Q$ we see that when $\ell = \ell_{reg}$ this requires $\ell(C) \geq 0$ for a cycle C. Here is an example where $\ell = \ell_{reg}$ and there are no negative cycles.

Electric cars: Suppose when we drive along edge e, $\ell(e)$ the amount of energy used is $\ell(e)$. This is normally positive, but when going down hill it can be negative. In this scenario, there can be no negative cycles under ℓ_{reg}.

Assume that the edges of D are $E = \{e_i = (x_i, y_i), i = 1, 2, \ldots, m\}$. Let $P_i, i = 1, 2, \ldots, n$ be a collection of paths, where $P_1 = (1)$ and P_i goes from 1 to i.

Lemma 1.2. *The following is a necessary and sufficient condition for $P_1, P_2, \ldots, P_n$ to be a collection of shortest paths with start vertex 1:*

$$\ell(P_y) \leq \ell(P_x * y) \text{ for all } (x, y) \in E. \tag{2}$$

Proof. It is clear that (2) is necessary. If it fails then $P_x * y$ is "shorter" than P_y.

Suppose that (2) holds. Let $P = (1 = x_0, x_1, x_2, \ldots, x_k = i)$ be a path from 1 to i. We show by induction on j that

$$\ell(P_{x_j}) \leq \ell(P(1, x_1, x_2, \ldots, x_j)). \tag{3}$$

Now when $j = 0$, both sides of (3) are zero. Then if it holds for some $j \geq 0$ then (2) and the inductive assumption imply that

$$\ell(P_{x_{j+1}}) \leq \ell(P_{x_j} * x_{j+1}) \leq \ell(P(1, x_1, \ldots, x_{j+1})).$$

Thus (2) is sufficient. □

Ford's Algorithm:
begin
for $i = 2, \ldots, n$, $d(i) \leftarrow \ell(1, i)$, $P_i \leftarrow (1, i)$;
    repeat;
    $flag \leftarrow 0$;
    for $i = 1, 2, \ldots, m$;
    begin
        if $\ell(P_{y_i}) > \ell(P_{x_i} * y_i)$ **then**;
        begin;
            $P_{y_i} \leftarrow (P_{x_i} * y_i)$; $flag \leftarrow 1$;
        end;
    end;
    until $flag = 0$;
    end
end

Lemma 1.3. *Ford's algorithm terminates after at most n rounds with a collection of shortest paths.*

Proof. If the algorithm terminates then because $flag = 0$ at this point, we have that (2) holds. Thus we have shortest paths.

We now argue that if the minimum number of arcs in a shortest path from 1 to i has ν_i edges then P_i is correct after ν_i rounds. We argue by induction. This is true for $i = 1$ and $\nu_i = 0$. Suppose that it is true for all i such that $\nu_i \leq \nu$ and that vertex j satisfies $\nu_j = \nu + 1$. Let $P = (1 = x_0, x_1, \ldots, x_{\nu+1} = j)$ be a shortest path from 1 to j. Then, by induction, after ν rounds P_{x_ν} is a shortest path from 1 to x_ν and then after one more round P_j is correct. $\square$

1.3 Digraphs without circuits

These are important, not least because they occur in Critical Path Analysis. Their application in this area involves computing longest paths.

1.3.1 Topological Ordering

Let the vertices of a digraph $D = ([n], E)$ be ordered $v_1, v_2, \ldots, v_n$. This ordering is *topological* if $(v_i, v_j) \in E$ implies that $i < j$.

Lemma 1.4. *Digraph D has a topological ordering if and only if D has no directed circuits.*

Proof. Suppose first that $v_1, v_2, \ldots, v_n$ is a topological ordering and that D has a directed cycle $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$. then we have $i_1 < i_2 < \cdots < i_k < i_1$, contradiction.

Conversely, suppose there are no directed circuits. Let $P = (x_1, x_2, \ldots, x_k)$ be a longest path in D. Then x_k is a *sink* i.e. there are no directed edges (x_k, y). (If $y \in X = \{x_1, x_2, \ldots, x_{k-1}\}$ then D contains a circuit. If $y \notin X$ then (P, x) is longer than P.)

To get a topological ordering, we let $v_n = x_k$ and inductively order the subgraph H induced by $[n] \setminus \{v_n\}$. This is a topological ordering. If $(v_i, v_j) \in E(H)$ then $i < j$ because H is toplologically ordered. Any other edge must be of the form (v_i, v_n). $\square$

To solve the longest path problem for paths starting at v_1, we take a topological ordering and then compute $d(v_1) = 0$ and then for $j \geq 2$,

$$d(v_j) = \max\left\{d(v_i) + \ell(v_i, v_j) : i < j \text{ and } (v_i, v_j) \in E\right\}. \tag{4}$$

Lemma 1.5. *Equation* (4) *computes the value of a longest path from v_1 to every other vertex.*

Proof. That $d(v_j)$ is correct follows by induction on j. It is trivially true for $j = 0$ and then for $j > 0$ we use the fact if $P = (x_1 = v_1, x_2, \ldots, x_k = v_j)$ is a longest path from v_1 to v_j then (i) $x_{k-1} = v_l$ for some $l < j$ and (ii) $(x_1, x_2, \ldots, x_{k-1})$ is a longest path from v_1 to v_l and (iii) $\ell(P) = d(v_l) + \ell(v_l, v_j)$. $\square$

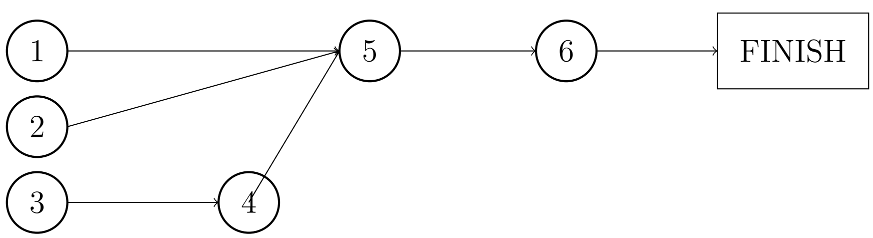
Critical Path Analysis: Imagine that a project consists of n activites.

Making a cup of tea:

1. Get a cup from the cupboard.
2. Get a tea bag.
3. Fill the kettle with water.
4. Boil the water.
5. Pour water into cup.
6. Allow to brew.

We define a digraph with n vertices, one for each activity and an edge (i, j) if (i) activity j cannot start until acivity i has been completed but (ii) only include (i, j) if it is not implied by a path (i, k, j). Each edge (i, j) has a length equal to the estimated duration of the activity i.

Tea Digraph:



Associate a time t_i to start activity i. Then t_i is the length of the longest path to vertex i. The estimated completion time of the project is then the length of the longest path to FINISH.

2 Assignment Problem

A *matching* M in a graph is a set of vertex disjoint edges. A vertex v is covered by M if there exists $e \in M$ such that $v \in e$. A matching M is *perfect* if every vertex of G coverd by M. For the complete bipartite graph $K_{A,B}$ on vertex set $A = \{a_i : i \in [n]\}, B = \{b_i : i \in [n]\}$, perfect matchings can be represented by permutations of n i.e $M = \left\{(a_i, b_{\pi(i)}) : i \in [n]\right\}$. Given a cost matrix $(c(i, j)$, the cost of a perfect matching $M = M(\pi)$ be given by

$$c(M) = \sum_{i=1}^{n} c(i, \pi(i)).$$

The assignment problem is that of finding a perfect matching of minimum cost.

2.1 Alternating paths

Given a matching M, a path $P = (e_1, e_2, \ldots, e_k)$ (as a sequence of edges) is *alternating* if the edges alternate between being in M and not in M.

An alternating path is *augmenting* if it begins and ends at uncovered vertices. If P is augmenting with respect to matching M, then $M' = M \oplus P$ is also a matching and $|M'| = |M| + 1$.

2.2 Successive shortest path algorithm

The algorithm produces a sequence $M_1, M_2, \ldots, M_n$ where M_k is a minimum cost matching from $[k]$ to $[k]$. It begins with $M_1 = (1, 1)$.

Suppose that $k > 1$ and that we have constructed $M_{k-1} = \left\{(a_i, b_{\pi(i)}) : i = 1, 2, \ldots, k-1\right\}$. The graph Γ_k is the complete graph K_{A_k,B_k}. The digraph $\vec{\Gamma}_k$ on vertex set $A_k = \{a_i : i \in [k]\}$, $B_k = \{b_i : i \in [k]\}$ is defined as follows. The directed edges are $X = \left\{(b_{\pi(i)}, a_i) : i \in [k-1]\right\}$ and $Y = \{(a_i, b_j) : i \in [k], j \in [k], j \neq \pi(i)\}$. The edge $(b_{\pi(i)}, a_i) \in X$ is given length $-c(i, \pi(i))$ and the edge $(i, j) \in Y$ is given length $c(i, j)$.

We observe the following:

- If M is a perfect matching of Γ_k then $M \oplus M_{k-1}$ consists of a collection $C_1, \ldots, C_p$ of vertex disjoint alternating cycles plus an augmenting path from a_k to b_k.
- $$c(M) - c(M_{k-1}) = \sum_{i=1}^{p} \ell(C_i) + \ell(P)$$
  where length ℓ is defined with respect to $\vec{\Gamma}_k$.
- $\ell(C_i) \geq 0$ for all i. Otherwise $M_{k-1} \oplus C_i$ is a matching of Γ_{k-1} with a cost $c(M_{k-1}) + \ell(C_i) < c(M_{k-1})$.

It follows from the above that to find a minimum cost matching of Γ_k, we should find a shotest path in $\vec{\Gamma}_k$ from a_k to b_k. Second, because $\vec{\Gamma}_k$ has no negative circuits, we can apply Ford's algorithm to find tihs path.

2.3 Linear Programming Solution – Hungarian Algorithm

Consider the linear program ALP:

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} x_{i,j} \tag{5}$$

Subject to

$$\sum_{j=1}^{n} x_{i,j} = 1 \qquad \text{for } i = 1, 2, \ldots, n. \tag{6}$$

$$\sum_{i=1}^{n} x_{i,j} = 1 \qquad \text{for } j = 1, 2, \ldots, n. \tag{7}$$

$$x_{i,j} \geq 0 \qquad \text{for } i, j = 1, 2, \ldots, n. \tag{8}$$

The assignment problem is the solution to ALP where we replace (8) by

$$x_{i,j} = 0 \text{ or } 1 \text{ for } i, j = 1, 2, \ldots, n. \tag{9}$$

This is because (6), (7) force the set $\{(i,j) : x_{i,j} = 1\}$ to be a perfect matching and (5) is then the cost of this matching.

In general replacing non-negativity constraints (8) by *integer* contraints (9) makes an LP hard to solve. Not however in this case.

The dual of ALP is the linear program DLP:

$$\text{Maximize} \quad \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \tag{10}$$

Subject to

$$u_i + v_j \leq c(i,j) \qquad \text{for } i, j = 1, 2, \ldots, n. \tag{11}$$

The *primal-dual* algorithm that we describe relies on *complimentary slackness* to find a solution.

Complimentary Slackness: If a feasible solution $\mathbf{x}$ to ALP and a feasible solution $\mathbf{u}$,$\mathbf{v}$, to DLP satisfy

$$x_{i,j} > 0 \text{ implies that } u_i + v_j = c(i,j). \tag{12}$$

then $\mathbf{x}$ solves ALP and $\mathbf{u}$,$\mathbf{v}$, solves DLP. For then

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{n} (c(i,j) - u_i - v_j) x_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} x_{i,j} - \left(\sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \right), \tag{13}$$

and the two solutions have the same objective value.

(We have used $\sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{i,j} = \sum_{i=1}^{n} u_i$, which follows from (6) etc.)

The steps of the Primal-Dual algorithm are as follows:

Step 1 Choose an initial dual feasible solution. E.g. $v_j = 0, j \in [n]$ and $u_i = \min_j c(i,j)$.

Step 2 Given a dual feasible solution, $\mathbf{u}$,$\mathbf{v}$, define the graph $K_{\mathbf{u},\mathbf{v}}$ to be the bipartite graph with vertex set A, B and an edge (i,j) whenever $u_i + v_j = c(i,j)$.

Step 3 Find a maximum size matching M in $K_{\mathbf{u},\mathbf{v}}$.

Step 4 If M is perfect then (12) holds and M provides a solution to the assignment problem.

Step 5 If M is not perfect, update **u,v** and go to Step 3.

To carry out Step 3, we proceed as follows:

Step 3a Begin with an arbitrary matching M of $K_{\mathbf{u},\mathbf{v}}$.

Step 3b Let A_U denote the set of vertices in A not covered by M.

Step 3c Let $\vec{K}_{\mathbf{u},\mathbf{v}}$ be the digraph obtained from $K_{\mathbf{u},\mathbf{v}}$ by orienting matching edges from B to A and other edges from A to B.

Step 3d Let A_M, B_M denote the set of vertices in A, B that are reachable by a path in $\vec{K}_{\mathbf{u},\mathbf{v}}$ from A_U. Such paths are necessarily alternating.

Step 3e If there is a vertex $b \in B_M$ that is not covered by M then there is an augmenting path P from some $a \in A_U$ to v. In this case we use P to consgtruct a matching M' with $|M'| > |M|$. We then go to Step 3b, with M replaced by M'. Otherwise, Step 3 is finished.

To carry out Step 5, we assume that we have finished Step 3 with M, A_M, B_M. We then let

$$\theta = \min\{c_{i,j} - u_i - v_j : a_i \in A_M, b_j \notin B_M\} > 0.$$

We know that $\theta > 0$. Otherwise, if a_i, b_j is the minimising pair, then we should have put $b_j \in B_M$.

We then amend **u,v** to $\mathbf{u}^*, \mathbf{v}^*$ via

$$u_i^* = \begin{cases} u_i + \theta & a_i \in A_M. \\ u_i & Otherwise. \end{cases} \text{ and } v_j^* = \begin{cases} v_j - \theta & j \in B_M. \\ v_j & Otherwise. \end{cases}$$

Observe the following:

1. $\mathbf{u}^*, \mathbf{v}^*$ is feasible for DLP. $u_i^* + v_j^* \leq u_i + v_j$ except for the case where $a_i \in A_M, b_j \notin B_M$ and θ is chosen so that the increase maintains feasiblity.
2. If $b \in B_M$ for the pair **u,v** then it will stay in B_M when we replace **u,v** by $\mathbf{u}^*, \mathbf{v}^*$. This is because there is a path $P = (a_{i_1} \in A_U, b_{i_1}, \ldots, a_{i_k}, b_{i_k} = b)$ such that each edge of P contains one vertex in A_M and one vertex in B_M. Hence the sum $u_i + v_j$ is unchanged for edges along P.
3. A vertex $b \notin B_M$ contained in a pair that defines θ will be in B_M when we replace **u,v** by $\mathbf{u}^*, \mathbf{v}^*$.

In summary: if we reach Step 4 with a perfect matching then we have solved ALP. After at most n changes of **u,v** in Step 5, the size of M increases by at least one. This is because updating **u,v** increases B_M by at least one. Thus the algorithm finishes in $O(n^4)$ time. ($O(n^3)$ time if done carefully.)

3 Branch and Bound

We consider the problem P_0:

$$\text{Minimize } f(x) \text{ subject to } x \in S_0.$$

Here S_0 is our set of feasible solutions and $f : S_0 \to \mathbb{R}$.

As we proceed in Branch-and-Bound we create a set of sub-problems $\mathcal{P}$. A sub-problem $P \in \mathcal{P}$ is defined by *the description of* a subset $S_P \subseteq S_0$. We also keep a *lower bound* b_P where

$$b_P \leq \min \{f(x) : \ x \in S_P\}.$$

At all times we act as if we have $x^* \in S_0$, some known feasible solution to P_0 and $v^* = f(x^*)$. If we do not actually have a solution x^* then we let $v^* = -\infty$. We will have a procedure BOUND that computes b_P for a sub-problem P. In many cases, BOUND *sometimes* produces a solution $x_P \in S_0$ and sometimes determines that $S_P = \emptyset$.

We initialize $\mathcal{P} = \{P_0\}$.

Branch and Bound:

Step 1 If $\mathcal{P} = \emptyset$ then x^* solves the problem.

Step 2 Choose $P \in \mathcal{P}$. $\mathcal{P} \leftarrow \mathcal{P} \setminus \{P\}$.

Step 3 **Bound:** Run BOUND(P) to compute b_P.

Step 4 If $S_P = \emptyset$ or $b_P \geq v^*$ then we consider P to be solved and go to Step 1.

Step 5 If BOUND generates $x_P \in S_0$ and $f(x_P) < v^*$ then we update, $x^* \leftarrow x_P, v^* \leftarrow f(x_P)$.

Step 6 **Branch:** Split P into a number of subproblems $Q_i, i = 1, 2, \ldots, \ell$, where $S_P = \bigcup_{i=1}^{\ell} S_{Q_i}$. And $S_{Q_i} \neq S_P$ is a strict subset for $i = 1, 2, \ldots, \ell$.

Step 7 $\mathcal{P} \leftarrow \mathcal{P} \cup \{Q_1, Q_2, \ldots, Q_\ell\}$.

Assuming S_0 is finite, this procedure will eventually terminate with $\mathcal{P} = \emptyset$. This is because the feasible sets S_P are getting smaller and smaller as we branch.

Most often the procedure BOUND has the following form: while it may be difficult to solve P directly, we may be able to find $T_P \supseteq S_P$ such that there is an efficient algorithm that determines whether or not $T_P = \emptyset$ and finds $\xi_P \in T_P$ that minimizes $f(\xi), \xi \in T_P$, if $T_P \neq \emptyset$. In this case, $b_P = f(\xi_P)$ and Step 5 is implemented if $\xi_P \in S_0$. We call the problem of minimizing $f(\xi), \xi \in T_P$, a *relaxed problem.*

Examples:

Ex. 1 Integer Linear Programming. Here S_P is the set of integer solutions and T_P is the set of solutions, if we ignore integrality. The procedure BOUND solves the linear program. If the solution ξ_P is not integral, we choose a variable x, whose value is $\zeta \notin \mathbb{Z}$ and form 2 sub-problems by adding $x \leq \lfloor z \rfloor$ to one and $x \geq \lceil z \rceil$ to the other.

Ex. 2 Traveling Salesperson Person Problem (TSP): Here S_P is the set of tours i.e. single directed cycles that cover all the vertices. We can take T_P to be the set of collections of vertex disjoint directed cycles that cover all the vertices. More precisely, to solve the TSP we must minimise $\sum_{i=1}^n C(I, \pi(i))$ as π ranges over all cyclic permutations. Our relaxation is to minimise $\sum_{i=1}^n C(I, \pi(i))$ as π ranges over all permutations, i.e. the assignment problem. We branch as follows. Suppose that the assignment solution consists of cycles $C_1, C_2, \ldots, C_k, k \geq 2$. Choose a cycle, C_1 say. Suppose that $C_1 = (v_1, v_2, \ldots, v_r)$ as a sequence of vertices. Then in Q_1 we disallow $\pi(v_1) = v_2$, in Q_2 we insist that $\pi(v_1) = v_2$, but that $\pi(v_2) \neq v_3$, in Q_3 we insist that $\pi(v_1) = v_2$, $\pi(v_2) = v_3$, but that $\pi(v_3) \neq v_4$ and so on.

Ex. 3 Implicit Enumeration: Here the problem is

$$\text{Minimize } \sum_{j=1}^n c_j x_j \text{ subject to } \sum_{j=1}^n a_{i,j} x_j \geq b_i,\ i \in [m], \quad x_j \in \{0,1\}, j \in [n].$$

A sub-problem is assciated with two sets $I, O \subseteq [n]$. This the sub-problem $P_{I,O}$ where we add the constraints $x_j = 1, j \in I, x_j = 0, j \in O$. We also check to see if $x_j = 1, j \in I, x_j = 0, j \notin I$ gives an improved feasible solution. As a bound $b_{I,O}$ we use $\sum_{j \notin O} \max\{c_j, 0\}$. To test feasibility we check that $\sum_{j \notin O} \max\{a_{i,j}, 0\} \geq b_i, i \in [m]$. To branch, we split $P_{I,O}$ into $P_{I \cup \{j\}, O}$ and $P_{I, O \cup \{j\}}$ for some $j \notin I \cup O$.

4 Matroids and the Greedy Algorithm

Given a *ground set* X, an *independence system* on X is collection of subsets $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$ such that

$$I \in \mathcal{I} \text{ and } J \subseteq I \text{ implies that } J \in \mathcal{I}. \tag{14}$$

Examples

Ex. 1 The set $\mathcal{M}$ of matchings of a graph $G = (V, X)$.

Ex. 2 The set of (edge-sets of) forests of a graph $G = (V, X)$.

Ex. 3 The set of *stable* sets of a graph $G = (X, E)$. We say that S is stable if it contains no edges.

Ex. 4 The set of solutions to the $\{0,1\}$-knapsack problem. Here we are given positive integers $w_1, w_2, \ldots, w_n, W$ and $X = [n]$ and $\mathcal{I} = \left\{S \subseteq [n] : \sum_{i \in S} w_i \leq W\right\}$.

Ex. 5 Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ be the columns of an $m \times n$ matrix $\mathbf{A}$. Then $X = [n]$ and $\mathcal{I} = \{S \subseteq [n] : \{\mathbf{c}_i, i \in S\}$ are linearly independent$\}$.

An independence system is a *matroid* if whenever $I, J \in \mathcal{I}$ with $|J| = |I| + 1$ there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$. Only Ex. 2 and 5 above are matroids. To check Ex. 5, let $\mathbf{A}_I$ be the $m \times |I|$ sub-matrix of $\mathbf{A}$ consisting of the columns in I. If there is no $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then $\mathbf{A}_J = \mathbf{A}_I \mathbf{M}$ for some $|I| \times |J|$ matrix. But then

$$|J| = rank(\mathbf{A}_J) \leq \min \{rank(\mathbf{A}_I), rank(\mathbf{M})\} \leq |I|,$$

contradiction.

To check Ex. 2 we can argue (exercise) that $I \subseteq E$ defines a forest if and only if the columns corresponding to I in the vertex-edge incidence matrix $\mathbf{M}_G$ are linearly independent.
($\mathbf{M}_G$ has a row for each vertex of G and a column for each edge of G. The column $\mathbf{c}_e, e = \{x, y\}$ has a one in row x and a -1 in row y and a zero in all other rows. It doesn't matter which of the two endpoints is viewed as x.)

4.1 Greedy Algorithm

Suppose that each $e \in E$ is given a weight w_e and that the weight $w(I)$ of an independent set I is given by $w(I) = \sum_{e \in I} c_e$. The problem we discuss is

$$\text{Maximize } w(I) \text{ subject to } I \in \mathcal{I}.$$

Greedy Algorithm:
begin
  Sort $E = \{e_1, e_2, \ldots, e_m\}$ so that $w(e_i) \geq w(e_{i+1})$ for $1 \leq i < m$;
  $S \leftarrow \emptyset$;
  for $i = 1, 2, \ldots, m$;
  begin
    if $S \cup \{e_i\} \in \mathcal{I}$ **then**;
    begin;
      $S \leftarrow S \cup \{e_i\}$;
    end;
  end;
end

Theorem 4.1. *The greedy algorithm finds a maximum weight independent set* **for all choices of** w *if and only if it is a matroid.*

Proof. Suppose first that the Greedy Algorithm always finds a maximum weight independent set. Suppose that $\emptyset \neq I, J \in \mathcal{I}$ with $|J| = |I| + 1$. Define

$$w(e) = \begin{cases} 1 + \frac{1}{2|I|} & e \in I. \\ 1 & e \in J \setminus I. \\ 0 & e \notin I \cup J. \end{cases}$$

If there does not exist $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$ then the Greedy Algorithm will choose the elements of I and stop. But I does not have maximum weight. Its weight is $|I| + 1/2 < |J|$. So if Greedy succeeds, then (**??**) holds.

Conversely, suppose that our independence system is a matroid. We can assume that $w(e) > 0$ for all $e \in E$. Otherwise we can restrict ourselves to the matroid defined by $\mathcal{I}' = \{I \subseteq E^+\}$ where $E^+ = \{e \in E : w(e) > 0\}$.

Suppose now that Greedy chooses $I_G = e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ where $i_t < i_{t+1}$ for $1 \leq t < k$. Let $I = e_{j_1}, e_{j_2}, \ldots, e_{j_\ell}$ be any other independent set and assume that $j_t < j_{t+1}$ for $1 \leq t < \ell$. We can assume that $\ell \geq k$, for otherwise we can add something from I_G to I to give it larger weight. We show next that $k = \ell$ and that $i_t \leq j_t$ for $1 \leq t \leq k$. This implies that $w(I_G) \geq w(I)$.

Suppose then that there exists t such that $i_t > j_t$ and let t be as small as possible for this to be true. Now consider $I = \{e_{i_s} : s = 1, 2, \ldots, t-1\}$ and $J = \{e_{j_s} : s = 1, 2, \ldots, t\}$. Now there exists $e_{j_s} \in J \setminus I$ such that $I \cup \{e_{j_s}\} \in \mathcal{I}$. But $j_s \leq j_t < i_t$ and Greedy should have chosen e_{j_s} before choosing $e_{i_{t+1}}$. Also, $i_k \leq j_k$ implies that $k = \ell$. Otherwise Greedy can find another element from $I \setminus I_G$ to add. □