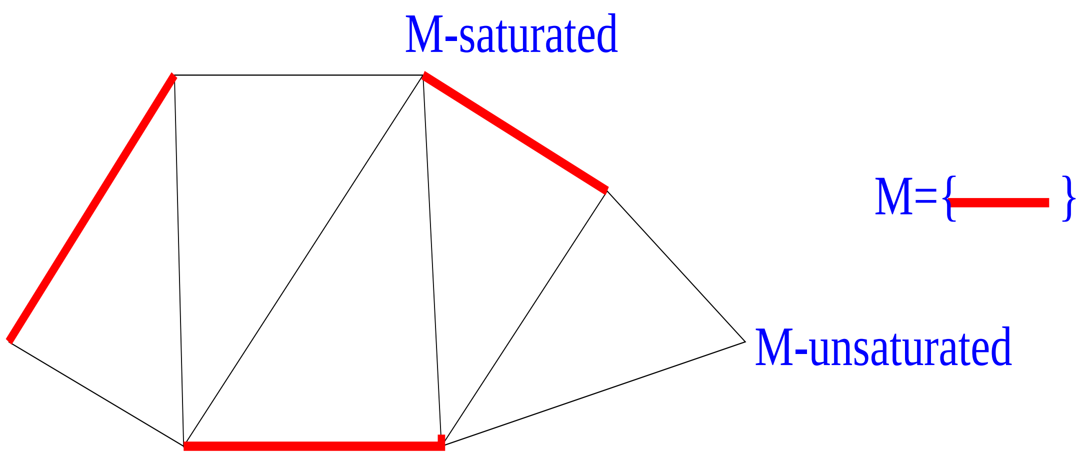
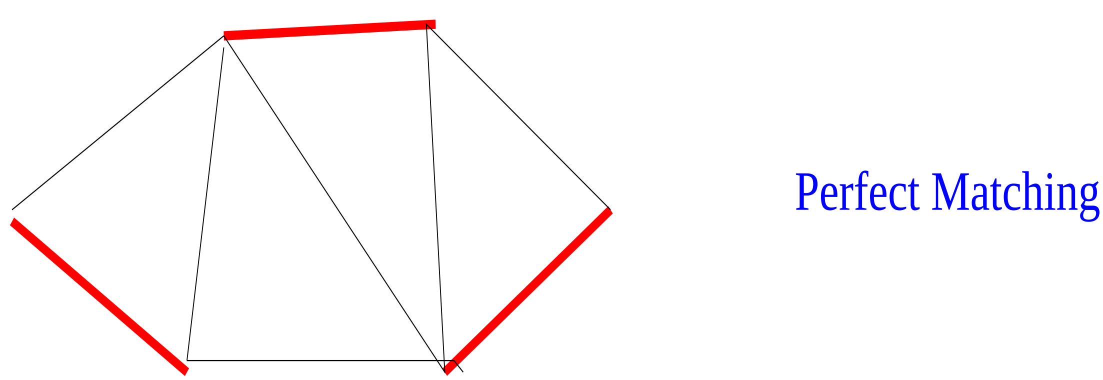
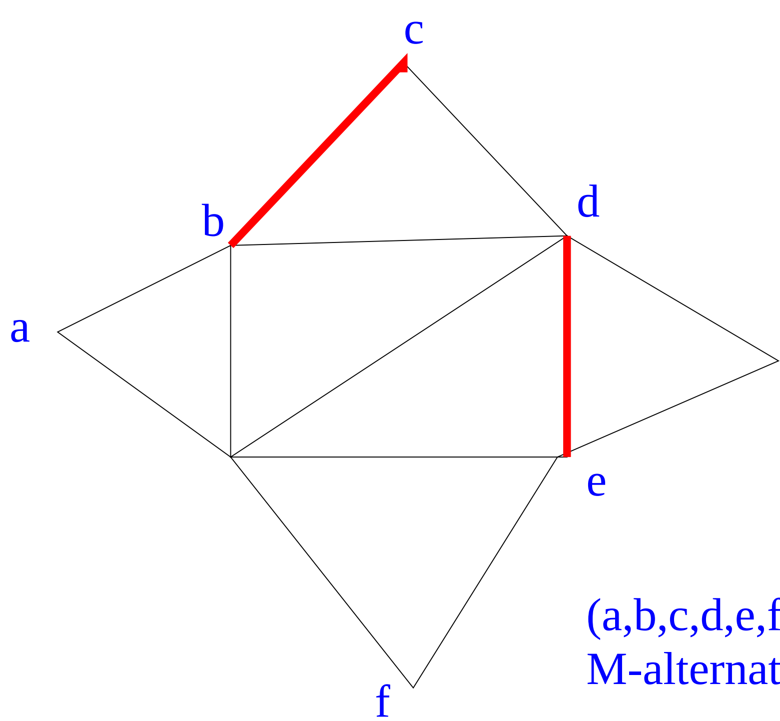
Matchings

A *matching* M of a graph $G = (V, E)$ is a set of edges, no two of which are incident to a common vertex.





M-alternating path



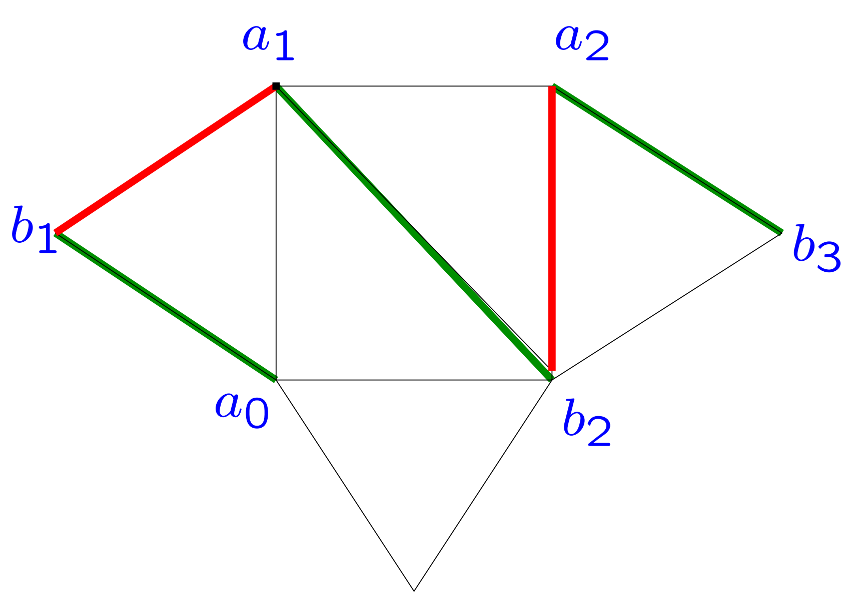
(a,b,c,d,e,f) is an M-alternating path

An M-alternating path joining 2 M-unsaturated vertices is called an M-augmenting path.

M is a *maximum* matching of G if no matching M' has more edges.

Theorem 1 *M is a maximum matching iff M admits no M-augmenting paths.*

Proof Suppose M has an augmenting path $P = (a_0, b_1, a_1, \ldots, a_k, b_{k+1})$ where $e_i = (a_{i-1}, b_i) \notin M$, $1 \leq i \leq k+1$ and $f_i = (b_i, a_i) \in M$, $1 \leq i \leq k$.



$$M' = M - \{f_1, f_2, \ldots, f_k\} + \{e_1, e_2, \ldots, e_{k+1}\}.$$

- $|M'| = |M| + 1$.

- M' is a matching

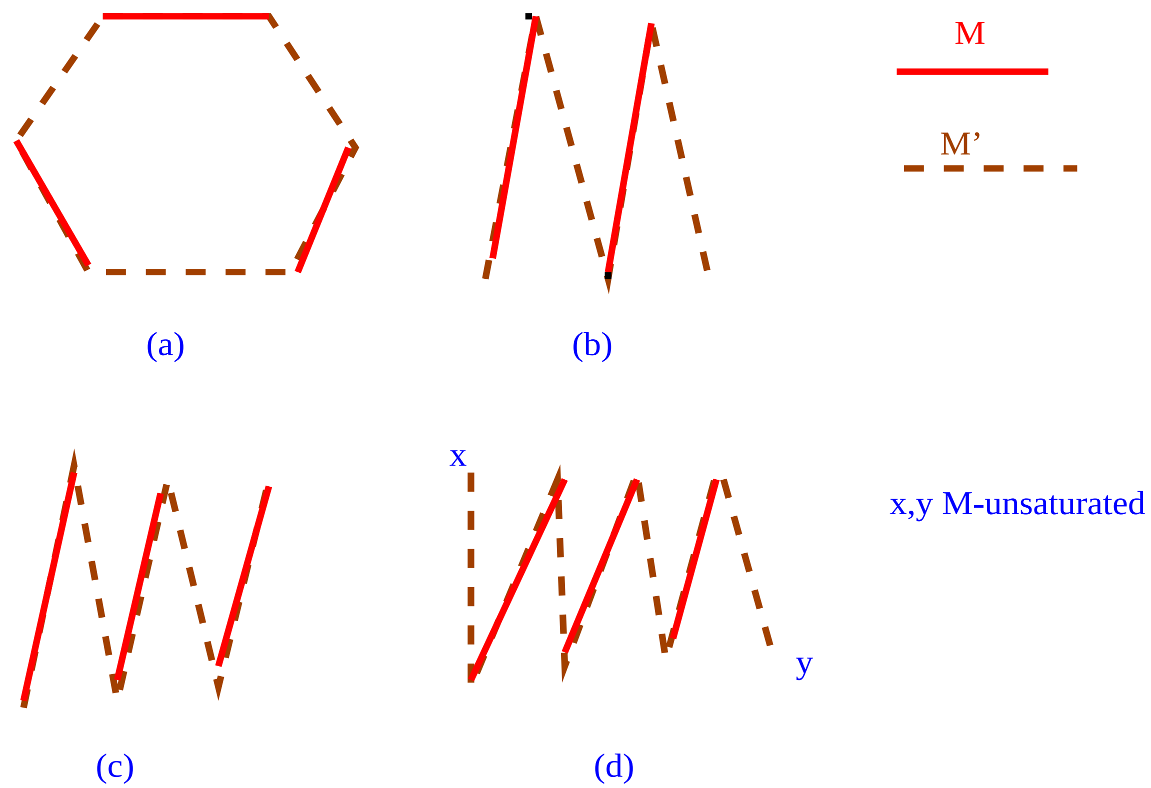
For $x \in V$ let $d_M(x)$ denote the degree of x in matching M, So $d_M(x)$ is 0 or 1.

$$d_{M'}(x) = \begin{cases} d_M(x) & x \notin \{a_0, b_1, \ldots, b_{k+1}\} \\ d_M(x) & x \in \{b_1, \ldots, a_k\} \\ d_M(x) + 1 & x \in \{a_0, b_{k+1}\} \end{cases}$$

So if M has an augmenting path it is not maximum.

Suppose M is not a maximum matching and $|M'| > |M|$. Consider $H = G[M \Delta M']$ where $M \Delta M' = (M \setminus M') \cup (M' \setminus M)$ is the set of edges in *exactly* one of M, M'.

Maximum degree of H is 2, at most 1 edge from M or M'. So H is a collection of vertex disjoint alternating paths and cycles.



(a) (b) (c) (d)

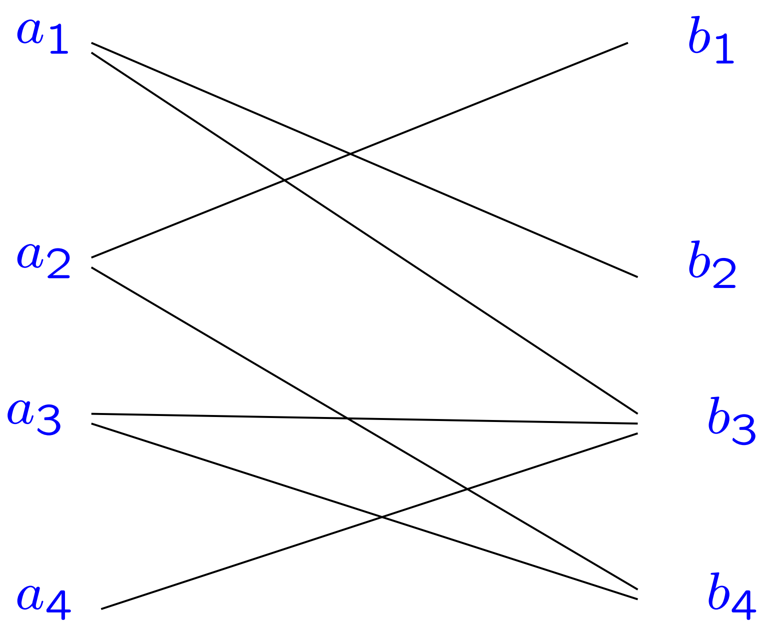
$|M'| > |M|$ implies that there is at least one path of type (d).

Such a path is M-augmenting $\square$

Bipartite Graphs

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.

For $S \subseteq A$ let $N(S) = \{b \in B : \exists a \in S, (a, b) \in E\}$.



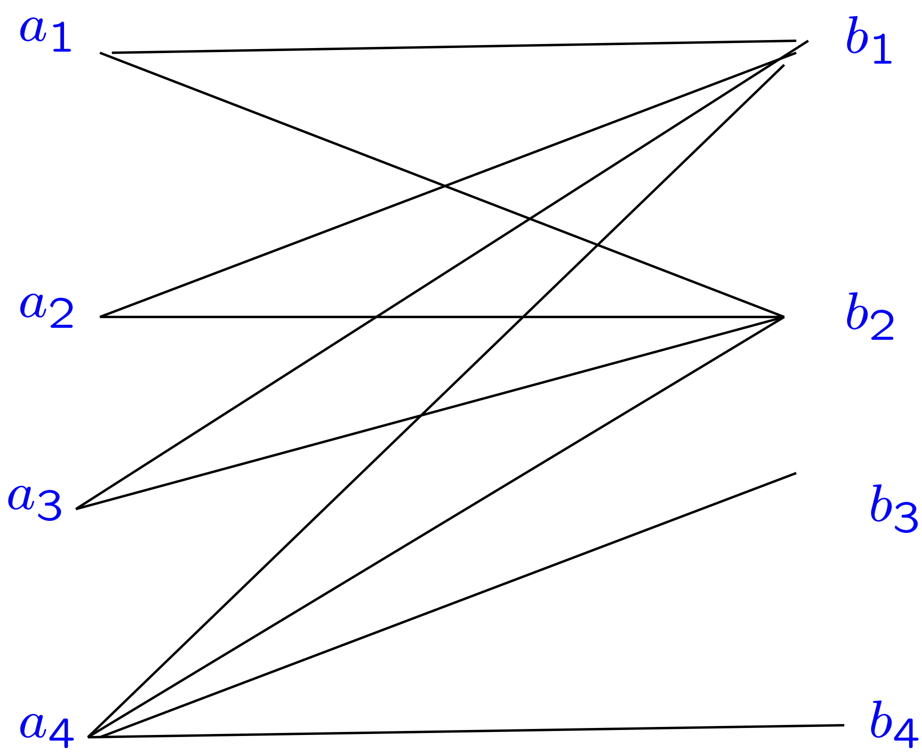
$$N(a_2, a_3) = \{b_1, b_3, b_4\}$$

Clearly, $|M| \leq |A|, |B|$ for any matching M of G.

Hall's Theorem

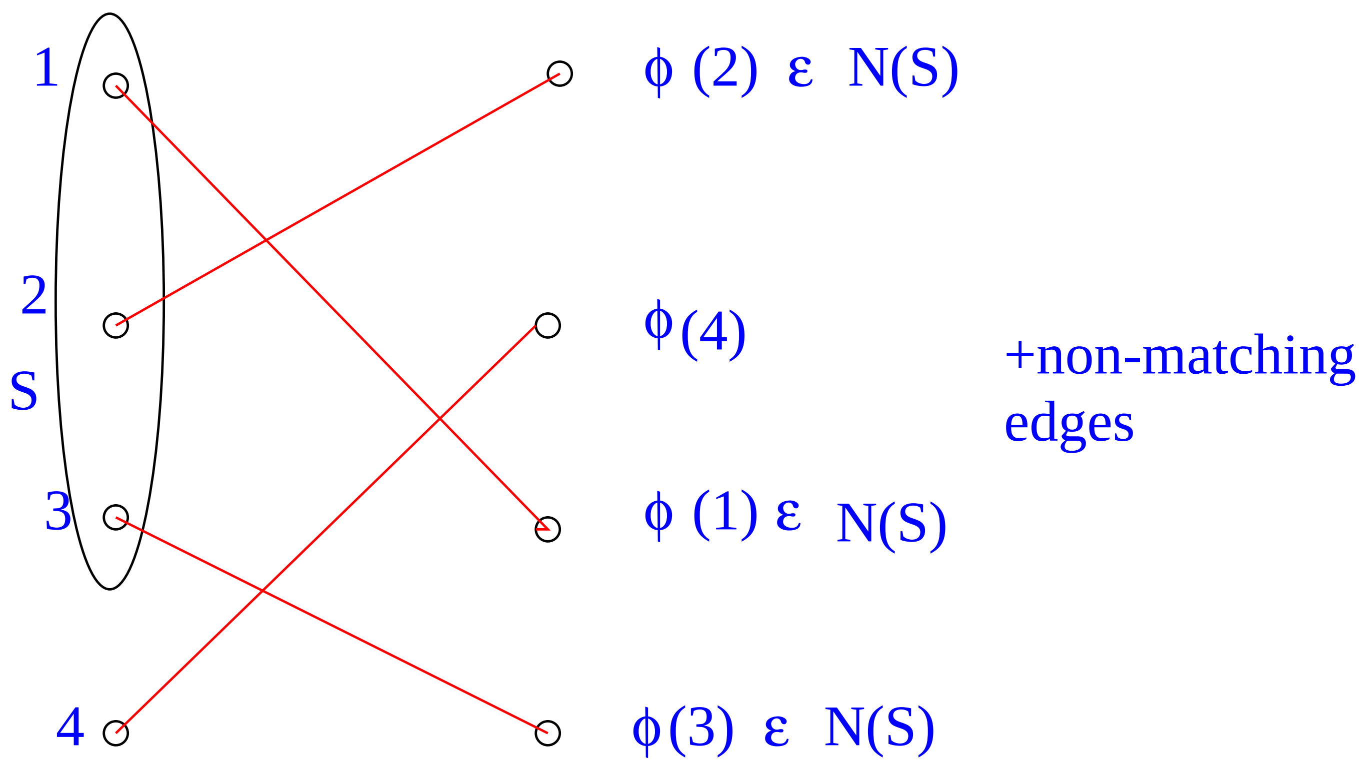
Theorem 2 *G contains a matching of size $|A|$ iff*

$$|N(S)| \geq |S| \qquad \forall S \subseteq A. \tag{1}$$



$N(\{a_1, a_2, a_3\}) = \{b_1, b_2\}$ and so at most 2 of a_1, a_2, a_3 can be saturated by a matching.

Only if: Suppose $M = \{(a, \phi(a)) : a \in A\}$ saturates A.



$$|N(S)| \geq |\{\phi(s) : s \in S\}| = |S|$$

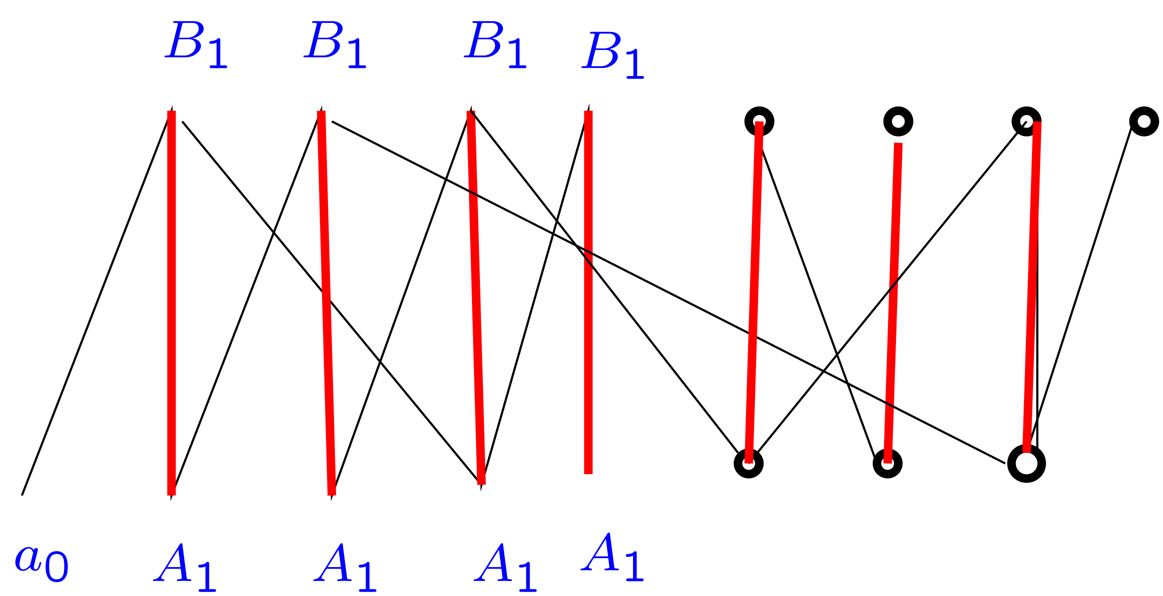
and so (1) holds.

If: Let $M = \{(a, \phi(a)) : a \in A'\}$ $(A' \subseteq A)$ is a maximum matching. Suppose $a_0 \in A$ is M-unsaturated. We show that (1) fails.

Let

$A_1 = \{a \in A$: such that a is reachable from a_0 by an M-alternating path.$\}$

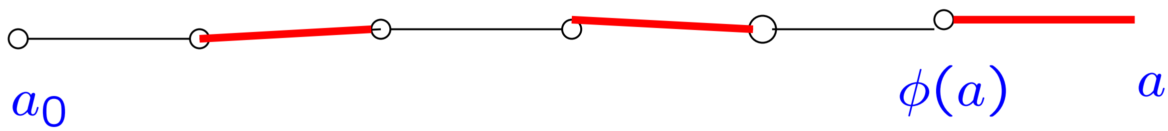
$B_1 = \{b \in B$: such that b is reachable from a_0 by an M-alternating path.$\}$



No $A_1 : B \setminus B_1$ edges

- B_1 is M-saturated else there exists an M-augmenting path.

- If $a \in A_1 \setminus \{a_0\}$ then $\phi(a) \in B_1$.

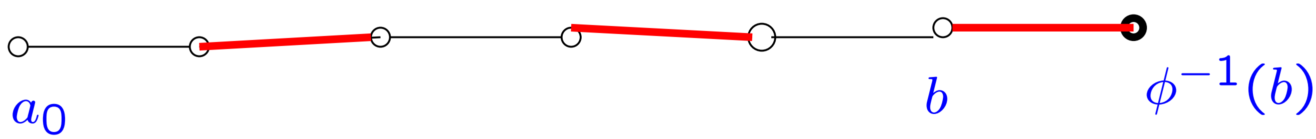


- If $b \in B_1$ then $\phi^{-1}(b) \in A_1 \setminus \{a_0\}$.

So

$$|B_1| = |A_1| - 1.$$

- $N(A_1) \subseteq B_1$



So

$$|N(A_1)| = |A_1| - 1$$

and (1) fails to hold.

Marriage Theorem

Theorem 3 *Suppose* $G = (A \cup B, E)$ *is* k*-regular.* *(*$k \geq 1$*) i.e.* $d_G(v) = k$ *for all* $v \in A \cup B$*. Then* G *has a perfect matching.*

Proof

$$k|A| = |E| = k|B|$$

and so $|A| = |B|$.

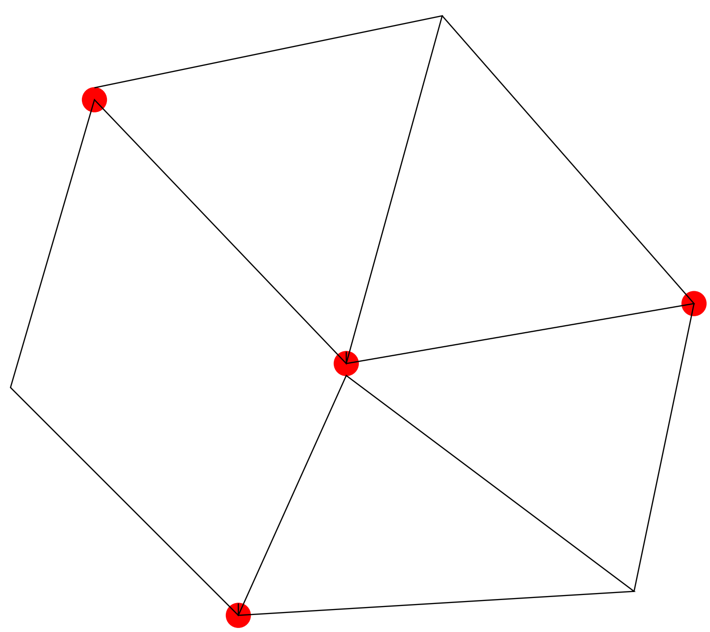
Suppose $S \subseteq A$. Let m be the number of edges incident with S. Then

$$k|S| = m \leq k|N(S)|.$$

So (1) holds and there is a matching of size $|A|$ i.e. a perfect matching.

Edge Covers

A set of vertices $X \subseteq V$ is a *covering* of $G = (V, E)$ if every edge of E contains at least one endpoint in X.



{ ● } is a covering

Lemma 1 *If X is a covering and M is a matching then $|X| \geq |M|$.*

Proof Let $M = \{(a_1, b_i) : 1 \leq i \leq k\}$. Then $|X| \geq |M|$ since $a_i \in X$ or $b_i \in X$ for $1 \leq i \leq k$ and $a_1, \ldots, b_k$ are distinct. □

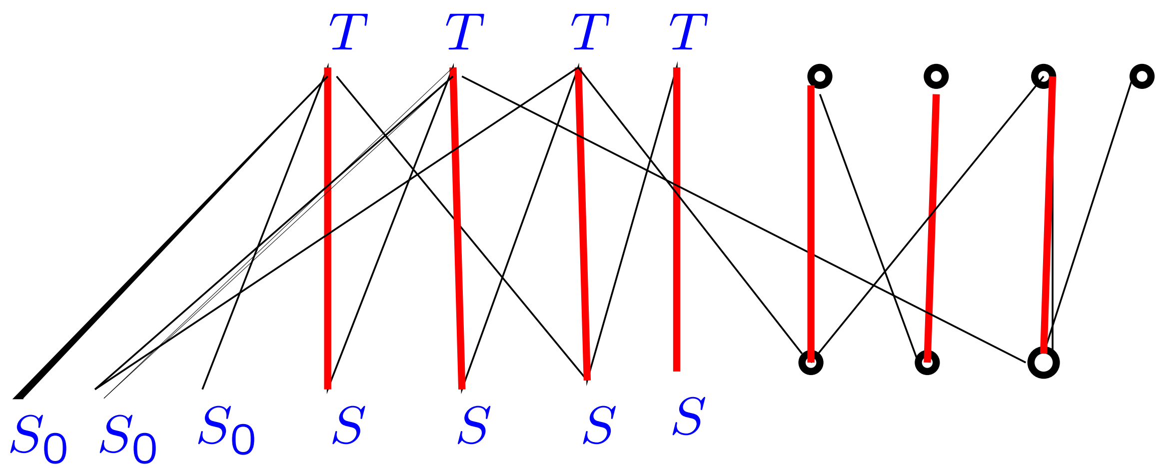
Konig's Theorem

Let $\mu(G)$ be the maximum size of a matching.
Let $\beta(G)$ be the minimum size of a covering.
Then

$$\mu(G) \leq \beta(G).$$

Theorem 4 *If G is* bipartite *then $\mu(G) = \beta(G)$.*

Proof Let M be a maximum matching.
Let S_0 be the M-unsaturated vertices of A.
Let $S \supseteq S_0$ be the A-vertices which are reachable from S by M-alternating paths.
Let T be the M-neighbours of $S \setminus S_0$.



Let $X = (A \setminus S) \cup T$.

- $|X| = |M|$.

$|T| = |S \setminus S_0|$. The remaining edges of M cover $A \setminus S$ exactly once.

- X is a cover.

There are no edges (x, y) where $x \in S$ and $y \in B \setminus T$. Otherwise, since y is M-saturated (no M-augmenting paths) the M-neighbour of y would have to be in S, contradicting $y \notin T$.

$\square$

Tutte's Theorem

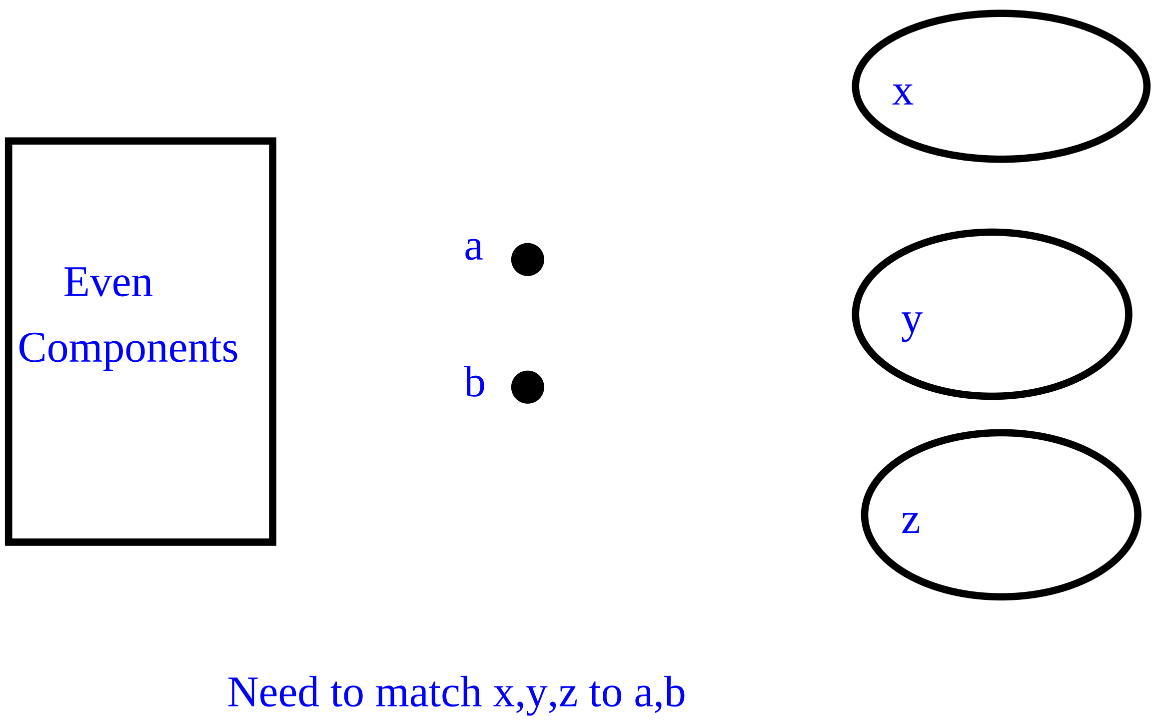
We now discuss arbitrary (i.e. non-bipartite) graphs.
For $S \subseteq V$ we let $o(G-S)$ denote the number of components of odd cardinality in $G-S$.

Theorem 5 *G has a perfect matching iff*

$$o(G-S) \leq |S| \qquad \textit{for all } S \subseteq V. \tag{2}$$

Proof We restrict our attention to simple graphs.

Only if:



Suppose $|S| = k$ and $O_1, O_2, \ldots, O_{k+1}$ are odd components of $G - S$. In any perfect matching of G, at least one vertex x_i of C_i will have to be matched outside O_i for $i = 1, 2, \ldots, k+1$. But then $x_1, x_2, \ldots, x_{k+1}$ will all have to be matched with S, which is impossible.

If: Suppose (2) holds and G has no perfect matching. Add edges until we have a graph G^* which satisfies

- G^* has no perfect matching.
- $G^* + e$ has a perfect matching for all $e \notin E(G^*)$.

Clearly,

$$o(G^* - S) \leq o(G - S) \leq |S| \qquad \text{for all } S \subseteq V. \tag{3}$$

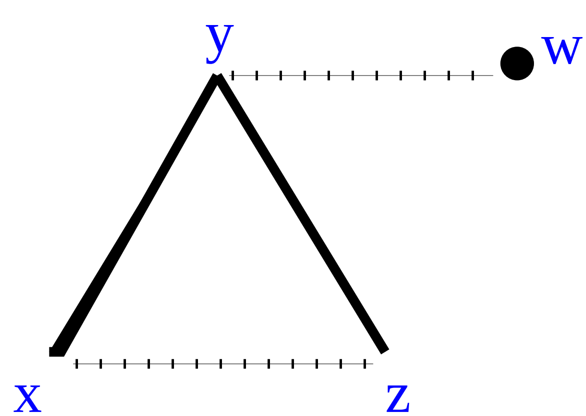
In particular, if $S = \emptyset$, $o(G^*) = 0$ and $|V|$ is even.

$$U = \{v \in V : \ d_{G^*}(v) = \nu - 1\}.$$

$U \neq V$ else G^* has a perfect matching.

Claim: $G^* - U$ is the disjoint union of complete graphs.

Suppose C is a component of $G^* - U$ which is not a clique. Then there exist $x, y, z \in C$ such that $xy, xz \in E(G^*)$ and $xz \notin E(G^*)$.
Take $x, z \in C$ at distance 2 in G^*.

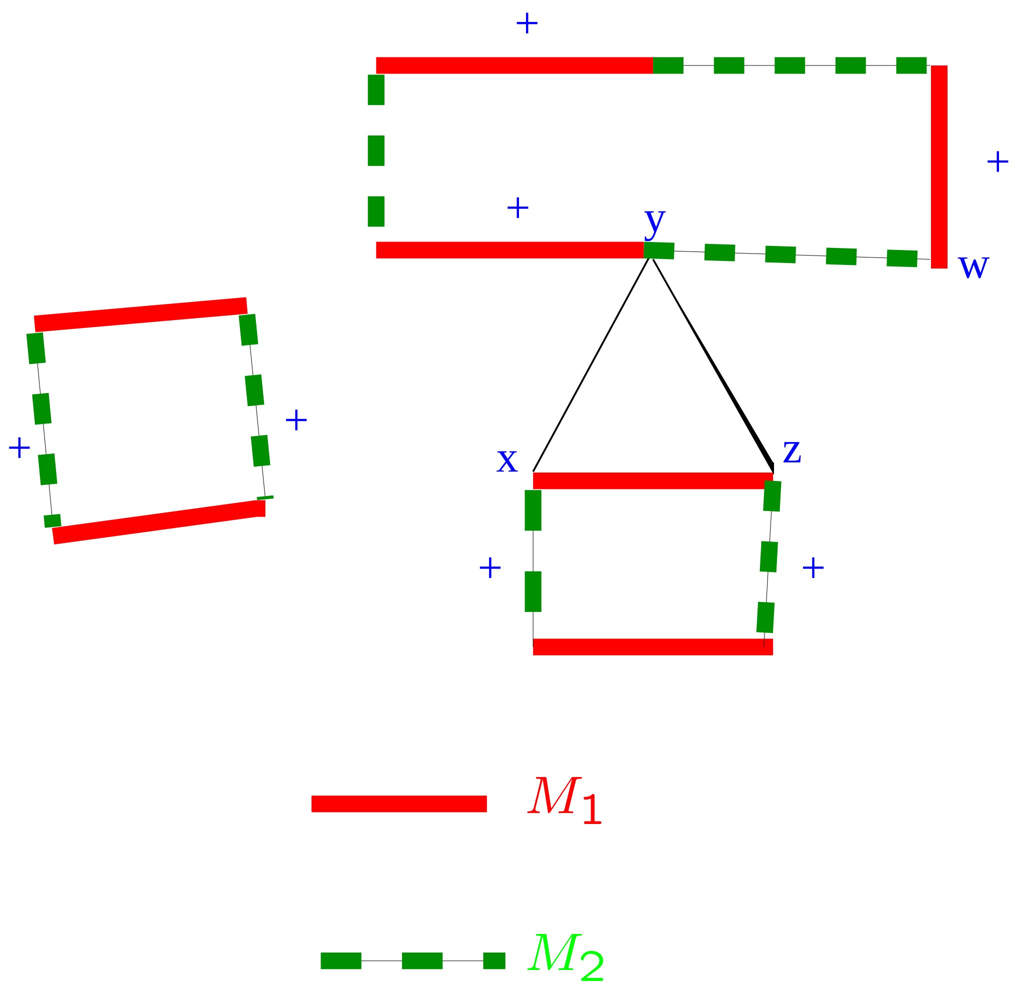


$y \notin U$ implies that there exists $w \notin U$ with $yw \notin E(G^*)$.

Let M_1, M_2 be perfect matchings in $G^* + xz, G^* + yw$ respectively.

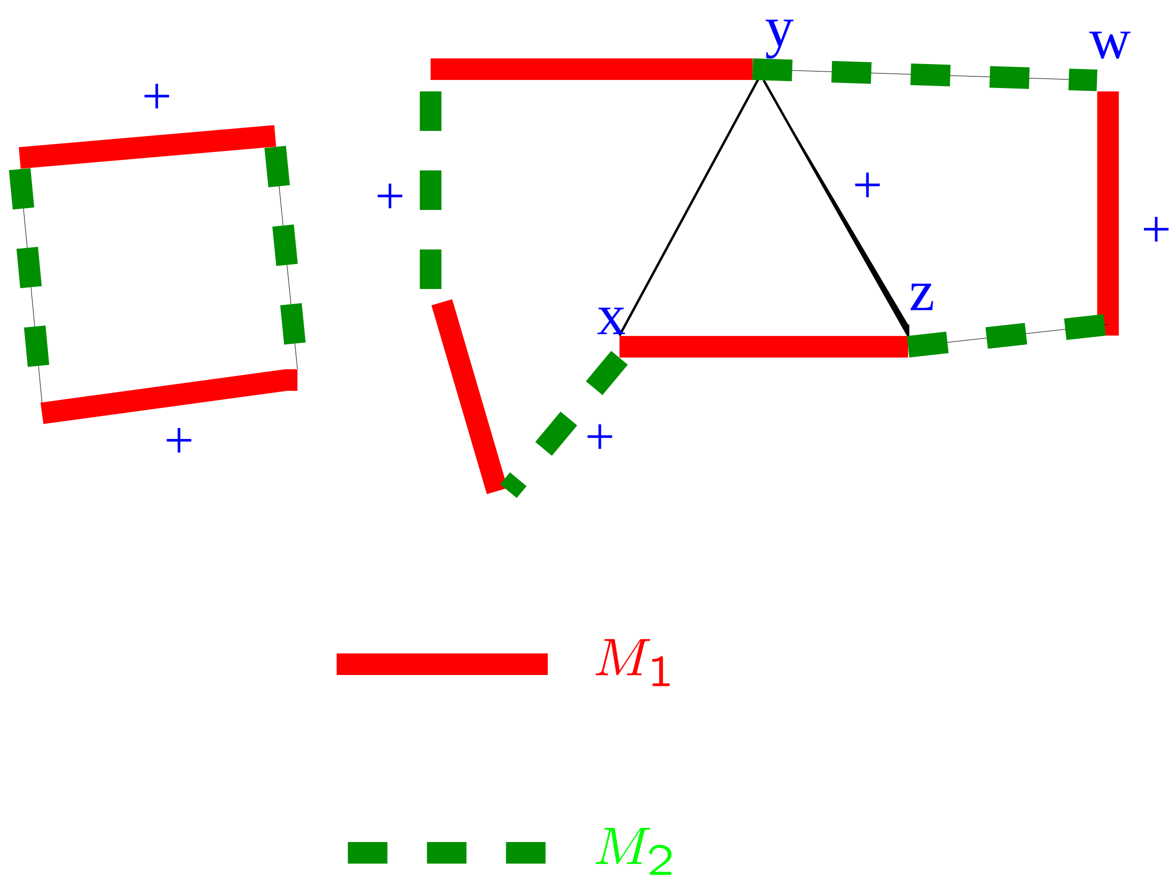
Let $H = M_1 \Delta M_2$. H is a collection of vertex disjoint even cycles.

Case 1: xz, yw are in different cycles of H.



+ edges form a perfect matching in G^* – contradiction.

Case 2: xz, yw are in same cycle of H.

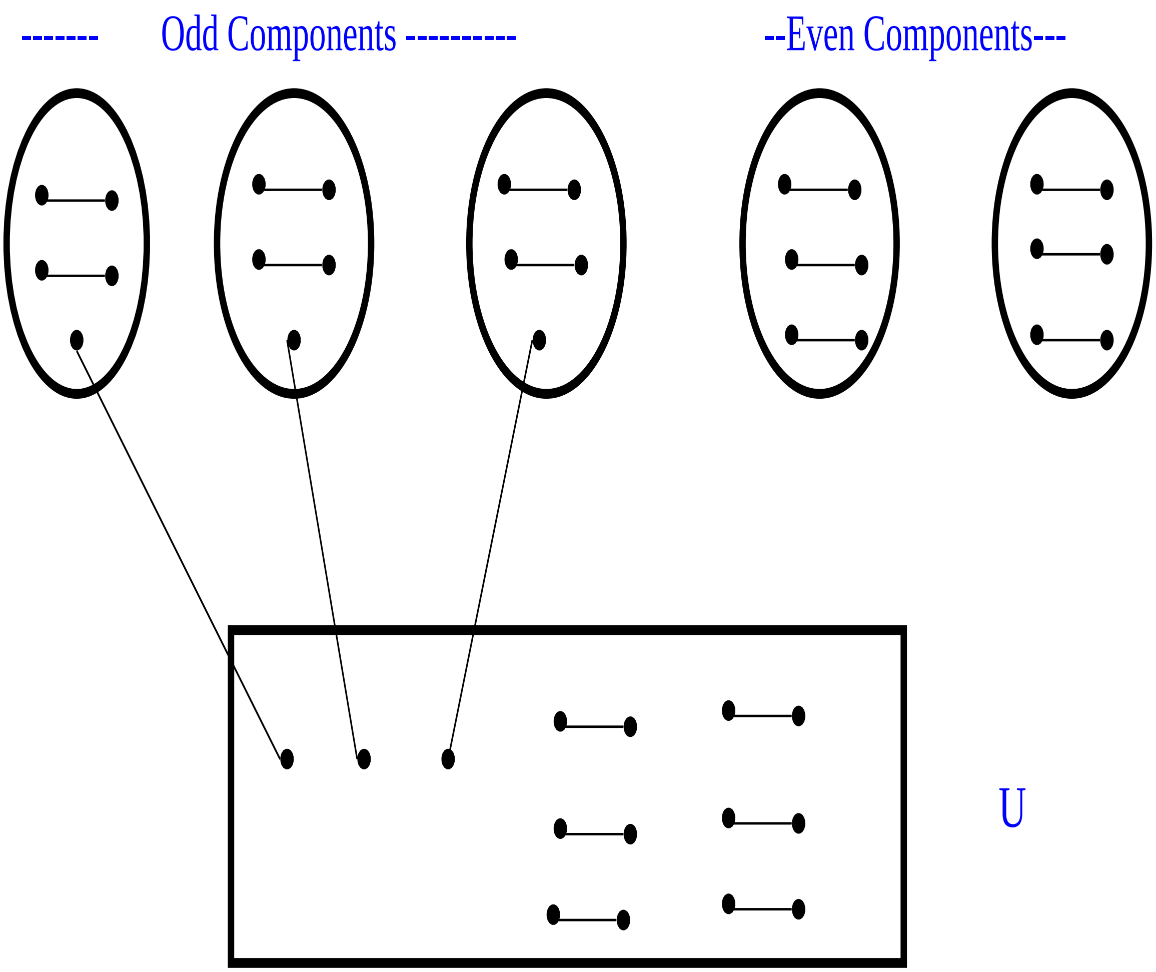


+ edges form a perfect matching in G^* – contradiction.

Claim is proved.

Suppose $G - U$ has ℓ odd components. Then

- $\ell \leq |U|$ from (3).
- $\ell = |U|$ mod 2, since $|V|$ is even.



G^* has a perfect matching – contradiction. $\square$

Petersen's Theorem

Theorem 6 *Every 3-regular graph without cut-edges contains a perfect matching.*

Proof Suppose $S \subseteq V$. Let $G - S$ have components $C_1, C_2, \ldots, C_r$ where $C_1, C_2, \ldots, C_\ell$ are odd.

m_i is the number of $C_i : S$ edges; $m_i \geq 2$.
n_i is the number of edges contained in C_i.

$$3|C_i| = m_i + 2n_i.$$

So m_i is odd for $1 \leq i \leq \ell$. Hence $m_i \geq 3$ for $1 \leq i \leq \ell$. Thus

$$3\ell \leq m_1 + m_2 + \cdots + m_\ell \leq 3|S|,$$

and (2) holds. □