SMALL SUBGRAPHS OF RANDOM GRAPHS

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- The expectation:

$$\mathbf{E}X = N(n,G)p^e = \binom{n}{v}\frac{v!}{aut(G)}p^e$$

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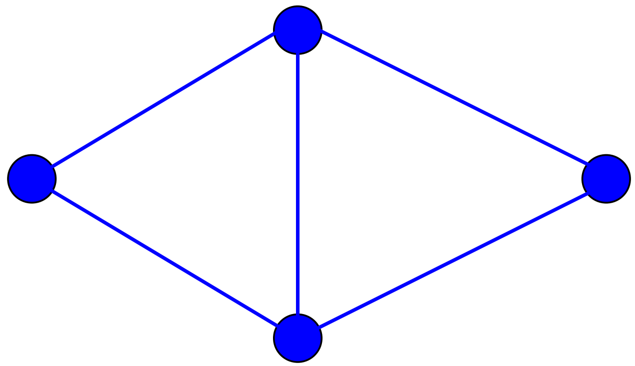

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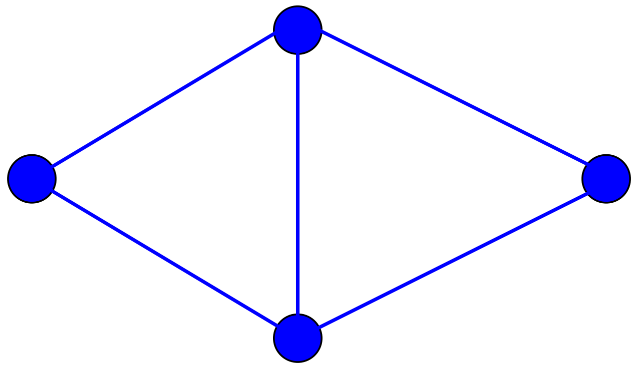

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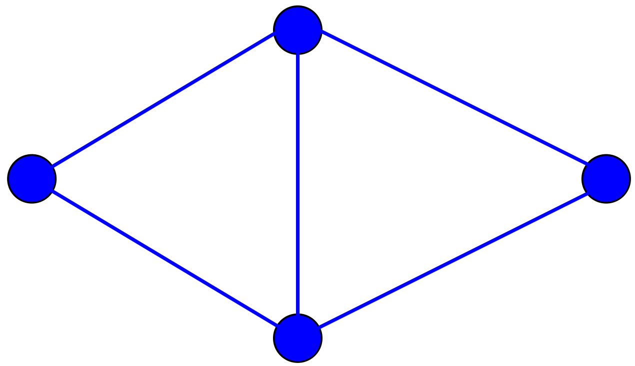

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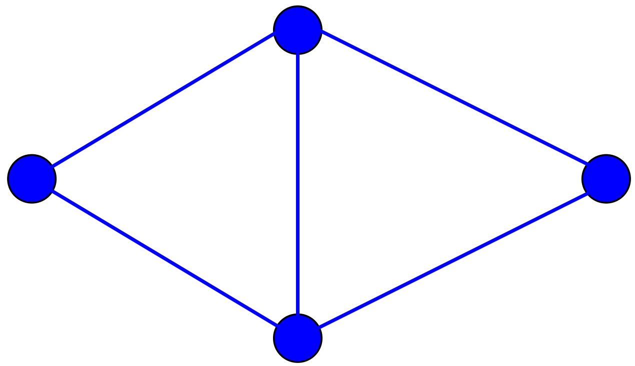

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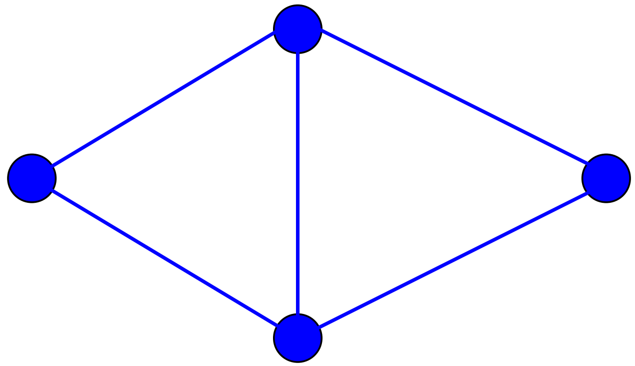

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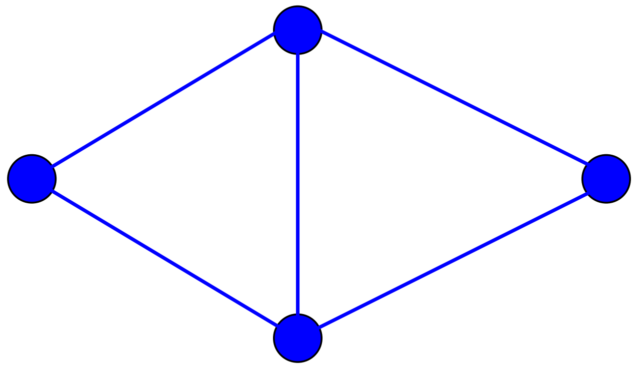

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The variance

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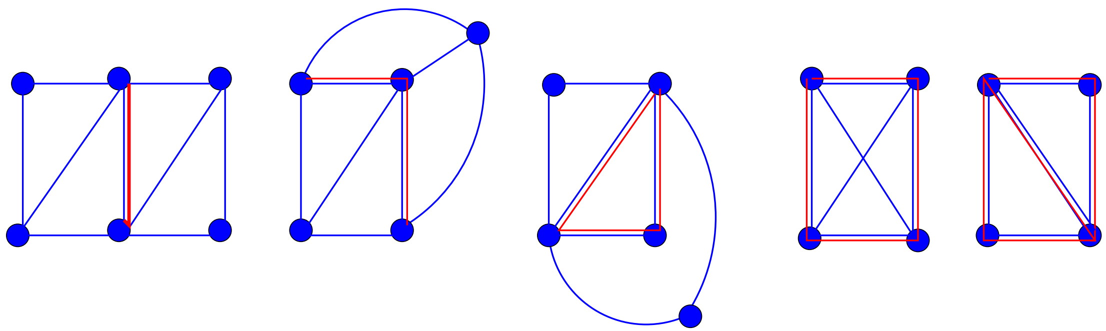

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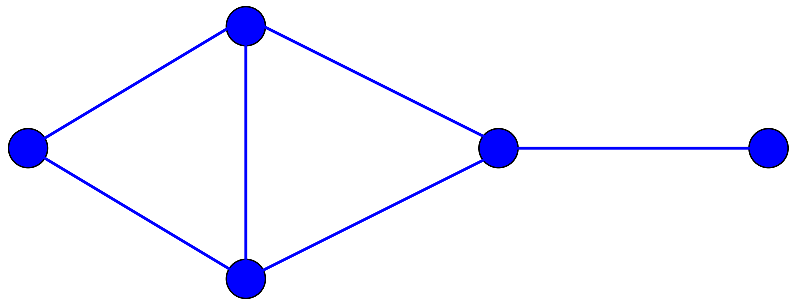

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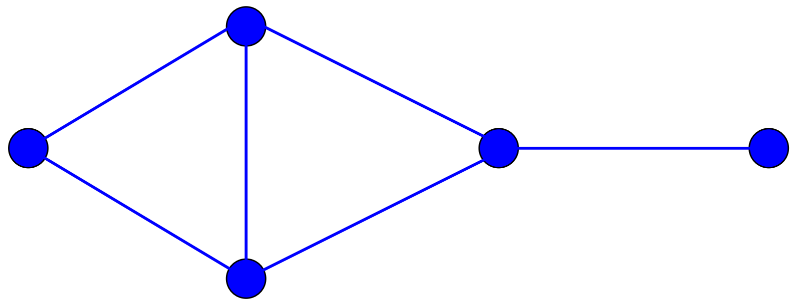

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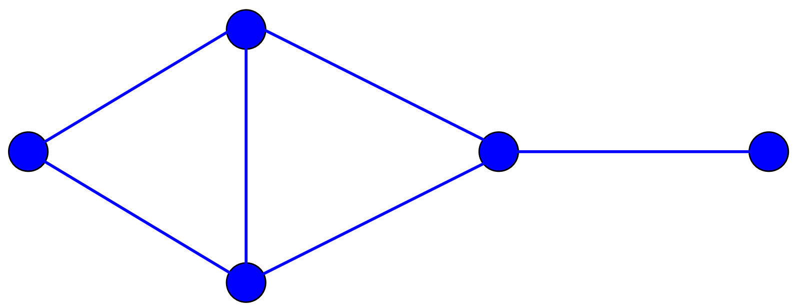

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$$\mathbf{P}(X_K > 0) \le \mathbf{P}(X_D > 0) = o(1)$$

Threshold - general case

In general, let

$$d_H = \frac{e_H}{v_H} \qquad \text{and} \qquad m_G = \max_{H \subseteq G} d_H .$$

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that is,

$$\mathbf{P}(X_G > 0) = \begin{cases} 0 \textit{ if } p \ll n^{-1/m_G} \\ 1 \textit{ if } p \gg n^{-1/m_G} \end{cases}$$

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$$\lim_{n\to\infty} \mathbf{P}(X_G = i) = e^{-\lambda}\frac{\lambda^i}{i!}.$$

The method of moments

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Note: $(X_G)_k$ counts ordered k-tuples of *distinct* copies of G in $G(n,p)$.

Proof for triangles

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where the sum splits over disjoint and not disjoint k-tuples.

Proof for triangles – cont.

$$\mathbf{E}'_k = \binom{n}{3,\ldots,3,n-3k} p^{3k} \sim \left(\frac{1}{6}n^3p^3\right)^k \sim (\mathbf{E}X_{K_3})^k$$

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$$\mathbf{E}(X_{K_3})_k = \mathbf{E}'_k + \mathbf{E}''_k \sim (\mathbf{E}X_{K_3})^k + O(p) \to \lambda^k$$

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Theorem (Ruciński (1988)) *For every graph G with $e_G > 0$,*

$$\frac{X_G - \mathbf{E}X_G}{\sqrt{var(X_G)}} \to \mathcal{N}(0,1) \text{ as } n \to \infty$$

if and only if $np^{m_G} \to \infty$ and $n^2(1-p) \to \infty$.

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Proof: By the method of moments (details omitted).

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Finally, with $\Psi_H = n^{v_H} p^{e_H}$ and $p = p(n) < c < 1$,

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By FKG, $\mathbf{P}(X = 0) \geq \exp\{-\mathbf{E}X/(1-p)\}$.

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Γ - finite set, Γ_p - a random binomial subset of Γ (each element included independently with probability p), $\mathcal{S}$ - family of subsets of Γ, for each $A \in \mathcal{S}$, I_A is the indicator of A in Γ_p,

$$X = \sum_{A \in \mathcal{S}} I_A$$

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Example. $\Gamma = \binom{[n]}{2}$, $\Gamma_p = G(n,p)$, $\mathcal{S}$ – all copies of G in K_n, $X = X_G$.

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Note: for disjoint A's, I_A's are independent and we get the (lower tail) Chernoff bound.

The rate of decay of $\mathrm{P}(X = 0)$

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By the chain formula

$$\mathbf{P}(X=0) = \mathbf{P}\left(\bigcap_{i=1}^{k} \bar{B}_i\right) = \prod_{i=1}^{k} \mathbf{P}\left(\bar{B}_i \mid \bigcap_{j=1}^{i-1} \bar{B}_j\right)$$

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Putting together

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Note that above is true for any subset of indices from $[k]$.

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The probabilistic method – cont.

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(recall $\Psi_H = n^{v_H} p^{e_H}$)

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Alon-Yuster, Ruciński: $p_0 = n^{-2/3}$ is the threshold for a perfect $(K_4 - K_{1,2})$-factor.

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Note: this threshold is sharp (Friedgut, Krivelevich, 1999)

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Switching to the model $G(n,M)$: about $c_0\binom{\binom{n}{2}}{(C/2)n^{4/3}}$ graphs with n vertices and $M = (C/2)n^{4/3}$ edges are such.

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Theorem (Friedgut,Rödl, Ruciński,Tetali (2005)) *The property* $G(n,p) \longrightarrow (K_3)^e_2$ *has a sharp threshold.*

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$F \longrightarrow (G)^e_r$ means that every r-coloring of the edges of F results in a monochromatic copy of G.

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Turán properties

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It is then possible to destroy all copies of G by deleting $o(n^2 p)$ edges – the Turán property does not hold for $G(n,p)$ in this case.

Turán properties for $G(n, p)$

Conjecture *For every* $\eta > 0$ *there is* $C > 0$ *such that if* $p \geq Cn^{-1/m_G^{(2)}}$ *then a.a.s. every subgraph of* $G(n, p)$ *with at least*

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(Frankl, Füredi, Gerke, Haxell, Kohayakawa, Kreuter, Łuczak, Rödl, Sabo, Schacht, Steger, Taraz, Vu, ...)

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Fractional independence number

α_G^* is the largest value of $\sum_v \alpha_v$ over all weightings $\alpha_v \in [0, 1]$ of $V(G)$ satisfying:

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Recall

$$\Psi_H = n^{v_H} p^{e_H}$$

Upper tail – new result

Theorem (Janson, Oleszkiewicz, Ruciński (2004))
For every graph G *and for every* $t > 1$ *there exist constants* $c(t, G) > 0$ *and* $C(t, G) > 0$ *such that for all* $n \geq v_G$ *and* $p \in (0, 1)$

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If $t\mathbf{E}X_G \leq N(n, G)$ then $tp^{e_G} \leq 1$, so $p \leq t^{-1/e_G} < 1$.

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Special cases: regular graphs, stars

Corollary *If* G *is a* k*-regular graph, then* $M_G^* = \Theta(n^2 p^k)$ *for all* $p \geq n^{-1/m_G} = n^{-2/k}$.

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Corollary *Let G be the k-armed star $K_{1,k}$, with $k \geq 1$, and assume $p \geq n^{-1/m_G} = n^{-1-1/k}$. Then*

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Special cases: paths

Corollary *Let P_k be the path on k vertices and assume $p \geq n^{-1/m_{P_k}} = n^{-1-1/(k-1)}$. Then, if $k \geq 3$ is odd,*

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Graphs with many phases

Let T^k be the tree obtained by taking k stars $K_{1,i}$, $i = 1, \ldots k$, and tying them up by merging one pendant vertex from each star into one vertex.

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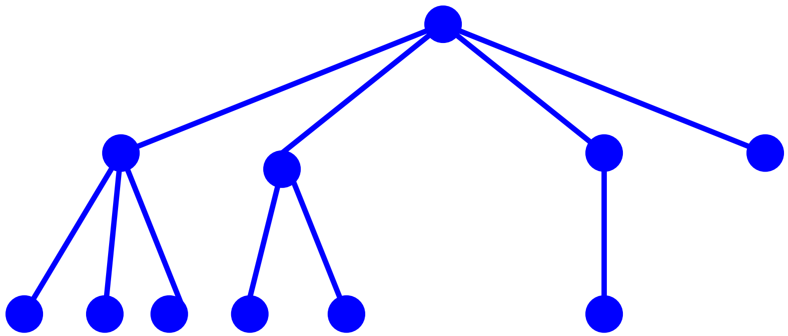

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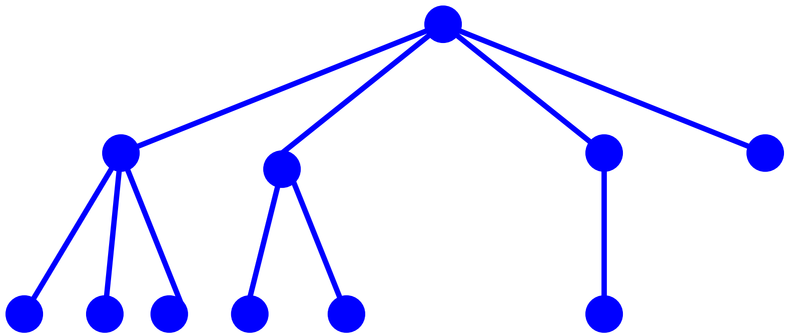

Proposition *For every $k \geq 2$, the graph T^k described above has $k + 1$ phases for the upper tail.*

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Finally,

$$\mathbf{P}(X_G \geq t\mathbf{E}X_G) \geq \mathbf{P}\left(G(n,p) \supseteq F\right) = p^{e_F} \geq p^m.$$

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$$\mathbf{P}(X_G \geq t\lambda_G) \leq t^{-m/2} = \exp\{-(m/2)\log t\} = \exp\{-cM_G^*\}$$

where $c = (c'/2)\log t$.

The mth moment

We will show by induction on m that

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$$\mathbf{E}(X_G^m) = \sum_{i_1,\ldots,i_m} \mathbf{E}(I_{i_1} \cdots I_{i_m}) = \sum_{i_1,\ldots,i_m} p^{e(G_{i_1} \cup \cdots \cup G_{i_m})}$$

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Let γ be the value of an optimal solution (x_v).

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Thus, neither end is sharp!