
Practice Exam 1

1. If \( A = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \) and \( AB = \begin{pmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{pmatrix} \), determine the first and second columns of \( B \).

**SOLUTION.** Since \( A \) has size 2 \( \times \) 2 and \( AB \) has size 2 \( \times \) 3, \( B \) has size 2 \( \times \) 3. Suppose \( B = \begin{pmatrix} a & c & * \\ b & d & * \end{pmatrix} \).

By the rule of matrix multiplication,

\[
AB = \begin{pmatrix} a - 2b & c - 2d & * \\ -2a + 5b & -2c + 5d & * \end{pmatrix}.
\]

Therefore, we have the following linear system:

\[
\begin{align*}
a - 2b &= -1 \\
-2a + 5b &= 6 \\
c - 2d &= 2 \\
-2c + 5d &= -9
\end{align*}
\]

Solving the system, we get \( a = 7, b = 4, c = -8, d = -5 \). So the first and second columns of \( B \) are \( \begin{pmatrix} 7 \\ 4 \end{pmatrix} \) and \( \begin{pmatrix} -8 \\ -5 \end{pmatrix} \).

2. Two matrices \( A \) and \( B \) are said to be similar, denoted \( A \sim B \), if there exists an invertible matrix \( P \) such that \( B = P^{-1}AP \). Prove:

(a) \( A \sim A \).

(b) If \( A \sim B \), then \( B \sim A \).

(c) If \( A \sim B \) and \( B \sim C \), then \( A \sim C \).

**PROOF.** (a) Taking \( P \) as the identity matrix \( I \), we have \( A = I^{-1}AI \). So \( A \sim A \).

(b) Since \( A \sim B \), there exists an invertible matrix \( P \) such that \( B = P^{-1}AP \). Notice that \( Q = P^{-1} \) is also invertible, and \( A = Q^{-1}BQ \). So \( B \sim A \).

(c) Since \( A \sim B \) and \( B \sim C \), there exist invertible matrices \( P \) and \( Q \) such that \( B = P^{-1}AP \), \( C = Q^{-1}BQ \). Notice that \( PQ \) is also invertible, and \( C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ) \). So \( A \sim C \).

3. (a) Explain why the inverse of a permutation matrix equals its transpose: \( P^{-1} = P^T \).

(b) If \( A^{-1} = A^T \), is \( A \) necessarily a permutation matrix? Give a proof or a counterexample to support your conclusion.

**SOLUTION.** (a) A permutation matrix is the product of a sequence of interchange elementary matrices. Suppose \( P = E_1E_2\cdots E_n \), each \( E_i \) interchanges some two rows of the identity matrix. It’s obvious that \( E_i \) is symmetric, so \( E_i^T = E_i \). Also, we have \( E_i^2 = I \) because applying the same interchange twice returns to the identity. Therefore,

\[
PP^T = (E_1E_2\cdots E_n)(E_1E_2\cdots E_n)^T = (E_1E_2\cdots E_n)(E_n^TE_{n-1}^T\cdots E_1^T) = (E_1E_2\cdots E_n)(E_nE_{n-1}\cdots E_1) = E_1\cdots E_{n-1}E_nE_{n-1}\cdots E_1,
\]

and

\[
P^TP = (E_1E_2\cdots E_n)^T(E_1E_2\cdots E_n) = (E_1E_2\cdots E_n)^T(E_nE_{n-1}\cdots E_1) = E_1\cdots E_{n-1}E_nE_{n-1}\cdots E_1 = \cdots = I.
\]

So \( P^{-1} = P^T \).
which implies $P^{-1} = P^T$.

(b) No. Let $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Then $A^{-1} = A^T = A$. But $A$ is not a permutation matrix, because it can’t be obtained by interchanging rows of the identity matrix. (If we look at $-1$ as a $1 \times 1$ matrix, it’s just an even simpler counterexample.)

4. Suppose $A$, $B$, and $X$ are $n \times n$ matrices with $A$, $X$, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B. \quad (1)$$

(a) Is $B$ invertible? Explain why.

(b) Solve (1) for $X$. If you need to invert a matrix, explain why that matrix is invertible.

**Solution.** (a) Yes. From (1) we get $B = X(A - AX)^{-1}$, the product of two invertible matrices $X$ and $(A - AX)^{-1}$. So $B$ is invertible.

(b) Since $(A - AX)^{-1}$, $X^{-1}$ and $B$ are invertible, from (1) we have

$$A - AX = ((A - AX)^{-1})^{-1} = (X^{-1}B)^{-1} = B^{-1}X,$$

or,

$$A = (A + B^{-1})X. \quad (2)$$

Since $X$ is invertible, $A + B^{-1} = AX^{-1}$, which is the product of two invertible matrices $A$ and $X^{-1}$. Therefore, $A + B^{-1}$ is invertible, and thus, from (2), we have $X = (A + B^{-1})^{-1}A$.

5. Find the determinant of the following Vandermonde matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$$

**Solution.** We reduce $A^T$ to an upper triangular matrix by elementary row operations.

$$A^T = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}$$

$$R_2/(b-a) \rightarrow \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b + a & b^2 + ba + a^2 \\ 0 & 1 & c + a & c^2 + ca + a^2 \\ 0 & 1 & d + a & d^2 + da + a^2 \end{pmatrix}$$

$$R_3/(c-a) \rightarrow \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b + a & b^2 + ba + a^2 \\ 0 & 0 & c - b & c^2 - b^2 + ca - ba \\ 0 & 1 & d + a & d^2 + da + a^2 \end{pmatrix}$$

$$R_4/(d-a) \rightarrow \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b + a & b^2 + ba + a^2 \\ 0 & 0 & c - b & c^2 - b^2 + ca - ba \\ 0 & 0 & 1 & c + b + a \end{pmatrix}$$

Therefore, $\det A = \det A^T = (b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$. 

\[ \square \]
6. When does the follow system have (i) a unique solution? (ii) no solution? (iii) infinitely many solutions?

\[
\begin{align*}
x + 3y - 2z &= 2 \\
y + z &= -5 \\
x + 2y - 3z &= a \\
-2x - 8y + 4z &= b
\end{align*}
\]

**Solution.** We reduce the augmented matrix to echelon form.

\[
\begin{pmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & 1 & -5 \\
1 & 2 & -3 & a \\
-2 & -8 & 4 & b
\end{pmatrix}
\]

\[
\overset{R_3\rightarrow R_3}{R_1-R_2} \rightarrow
\overset{R_4+2R_1}{R_4+2R_2} \rightarrow
\begin{pmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & 1 & -5 \\
0 & -1 & -1 & a - 2 \\
0 & 0 & 2 & b + 4
\end{pmatrix}
\]

\[
\overset{R_3\rightarrow R_3}{R_3-R_1} \rightarrow
\begin{pmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & 1 & -5 \\
0 & 0 & 2 & b - 6 \\
0 & 0 & 0 & a - 7
\end{pmatrix}
\]

Now we see each column contains a pivot, so the system can’t have infinitely many solutions. When \(a - 7 = 0\), or \(a = 7\), the system is consistent and has a unique solution. When \(a \neq 7\), the system is inconsistent and has no solution.

7. If \(A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\), show that \(K = AA^T\) is well-defined, symmetric matrix. Find the \(LDL^T\) factorization of \(K\).

**Solution.** \(A\) has size \(2 \times 3\), and \(A^T\) has size \(3 \times 2\). So \(K = AA^T\) is well-defined. \(K\) is symmetric because \(K^T = (AA^T)^T = (A^T)^T A^T = AA^T = K\). Symmetry can also be seen by direct computation:

\[
K = AA^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}
\]

To find the \(LDL^T\) factorization, we apply Gaussian method to \(K\):

\[
\begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix} \overset{R_2-(16/7)R_1}{\rightarrow} \begin{pmatrix} 14 & 32 \\ 0 & 27/7 \end{pmatrix} = U,
\]

\[
\overset{R_2\rightarrow R_2}{R_1-R_2} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 16/7 & 1 \end{pmatrix} = L,
\]

and \(D = \begin{pmatrix} 14 & 0 \\ 0 & 27/7 \end{pmatrix}\), the diagonal part of \(U\). Thus the \(LDL^T\) factorization of \(K\) is

\[
\begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 16/7 & 1 \end{pmatrix} \begin{pmatrix} 14 & 0 \\ 0 & 27/7 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}^T.
\]
8. Use the Gauss-Jordan method to find the inverse of the following complex matrix:

\[
\begin{pmatrix}
0 & 1 & -i \\
i & 0 & -1 \\
-1 & i & 1
\end{pmatrix}
\]

**Solution.**

\[
\begin{pmatrix}
0 & 1 & -i & 1 & 0 & 0 \\
i & 0 & -1 & 0 & 1 & 0 \\
-1 & i & 1 & 0 & 0 & 1
\end{pmatrix}
\]

\(R_{1} \rightarrow R_{3}\),

\[
\begin{pmatrix}
-1 & i & 1 & 0 & 1 & 0 \\
i & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & -i & 1 & 0 & 0
\end{pmatrix}
\]

\(R_{1} \times (-1)\),

\[
\begin{pmatrix}
1 & -i & -1 & 0 & 0 & -1 \\
i & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & -i & 1 & 0 & 0
\end{pmatrix}
\]

\(R_{2} \rightarrow \overline{R_{1}}\),

\[
\begin{pmatrix}
1 & -i & -1 & 0 & 0 & -1 \\
i & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & -i & 1 & 0 & 0
\end{pmatrix}
\]

\(R_{2} \times (-1)\),

\[
\begin{pmatrix}
1 & -i & -1 & 0 & 0 & -1 \\
i & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & -i & 1 & 0 & 0
\end{pmatrix}
\]

\(R_{1} \rightarrow R_{3}\),

\[
\begin{pmatrix}
1 & 0 & i & 0 & -i & 0 \\
0 & 1 & i & 0 & -1 & -i \\
0 & 0 & -1 & 0 & 1 & i
\end{pmatrix}
\]

\(R_{3} \rightarrow R_{2}\),

\[
\begin{pmatrix}
1 & 0 & i & 0 & -i & 0 \\
0 & 1 & i & 0 & -1 & -i \\
0 & 0 & -1 & 1 & 1 & i
\end{pmatrix}
\]

\(R_{2} \rightarrow (1-i)R_{3}\),

\[
\begin{pmatrix}
1 & 0 & i & 0 & -1 \\
0 & 1 & 0 & 1-i & -i & 1 \\
0 & 0 & 1 & -1 & -1 & -i
\end{pmatrix}
\]

So,

\[
\begin{pmatrix}
0 & 1 & -i \\
i & 0 & -1 \\
-1 & i & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
i & 0 & -1 \\
1-i & -i & 1 \\
-1 & 1 & -i
\end{pmatrix}.
\]