Exam 2 Solutions

1. (18 points) True or False. (Don’t need to justify)

(a) The set of all vectors of the form \( \begin{pmatrix} 3a + b \\ 4 \\ a - 5b \end{pmatrix} \), where \( a, b \) represent arbitrary real numbers, is a vector space.
   
   Solution. False. The zero vector \( \mathbf{0} \) doesn’t belong to the set.

(b) \( \mathbb{R}^2 \) is a subspace of \( \mathbb{R}^3 \).
   
   Solution. False. \( \mathbb{R}^2 \) is not a subset of \( \mathbb{R}^3 \).

(c) If \( U \) is the echelon form of \( A \), then \( \text{Col} U = \text{Col} A \).
   
   Solution. False. A simple counter-example: \( A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, U = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

(d) The kernel of an \( m \times n \) matrix is in \( \mathbb{R}^m \).
   
   Solution. False. The kernel of an \( m \times n \) matrix is in \( \mathbb{R}^n \), while the range is in \( \mathbb{R}^m \).

(e) A positive definite quadratic form \( q \) satisfies \( q(\mathbf{x}) > 0 \) for all \( \mathbf{x} \) in \( \mathbb{R}^n \).
   
   Solution. False. \( q(\mathbf{0}) = 0 \).

(f) If \( A \) is a square matrix, \( \mathbf{u} \in \text{Col} A, \mathbf{v} \in \ker A \), then \( \mathbf{u} \perp \mathbf{v} \).
   
   Solution. False. For example, \( A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \text{Col} A, \mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \in \ker A \). But \( \mathbf{u} \cdot \mathbf{v} \neq 0 \), so, \( \mathbf{u} \not\perp \mathbf{v} \). \( \square \)

2. (15 points) Find the value(s) of \( h \) for which the vectors \( \begin{pmatrix} 1 & -2 & 3 \\ 5 & -9 & h \end{pmatrix} \) are linearly dependent.

Solution. Let matrix \( A \) have the three vectors as its columns. Apply Gaussian to reduce \( A \) in the echelon form:

\[
\begin{pmatrix} 1 & -2 & 3 \\ 5 & -9 & h \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & h - 15 \end{pmatrix}
\]

We see that \( A \) always has only 2 pivots regardless of the value of \( h \). Therefore, the three vectors are linearly dependent for any real number \( h \). \( \square \)

3. (18 points)

(a) Prove that the only element \( \mathbf{w} \) in an inner product space \( V \) that is orthogonal to every vector is the zero vector \( \mathbf{w} = \mathbf{0} \).

Proof. Since \( \mathbf{w} \) is orthogonal to every vector, it’s orthogonal to itself, namely \( \langle \mathbf{w}, \mathbf{w} \rangle = 0 \). By positivity, \( \mathbf{w} = 0 \). \( \square \)
(b) Prove that \( \|w\| \leq \|v\| + \|v + w\| \) for any \( v, w \in V \).

**Proof.** By the triangle inequality,

\[
\|w\| = \|(v + w) + (-v)\| \leq \|v + w\| + \|-v\| = \|v\| + \|v + w\|.
\]

\( \square \)

4. (16 points) Show that \( v_1 = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \) and \( w_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} -6 \\ 2 \\ -1 \end{pmatrix} \) are two bases for the same two-dimensional subspace \( V \in \mathbb{R}^3 \).

**Proof.** Let \( A = (v_1 \ v_2), B = (w_1 \ w_2) \). We are to characterize \( \text{Col} A \) and \( \text{Col} B \). First consider the system \( Ax = b \), where the right hand side will remain unspecified for the moment. Apply Gaussian to the augmented matrix:

\[
\begin{pmatrix}
4 & 2 & b_1 \\
1 & -2 & b_2 \\
3 & -1 & b_3 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -2 & b_2 \\
0 & 4 & 2b_1 \\
0 & 3 & -1b_3 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -2 & b_2 \\
0 & 10 & b_1 - 4b_2 \\
0 & 5 & b_3 - 3b_2 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -2 & b_2 \\
0 & 10 & 0 \\
0 & 5 & -b_1 - 2b_2 + b_3 \\
\end{pmatrix}
\]

Hence, \( \text{Col} A = \{(b_1, b_2, b_3)^T | -\frac{1}{2}b_1 - b_2 + b_3 = 0\} \), a two-dimensional subspace of \( \mathbb{R}^3 \). Similarly,

\[
\begin{pmatrix}
2 & -6 & b_1 \\
0 & 2 & b_2 \\
1 & -1 & b_3 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & b_3 \\
0 & 2 & 2b_2 \\
0 & -6 & b_1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & b_3 \\
0 & 2 & 0 \\
0 & 0 & b_1 + 2b_2 - 2b_3 \\
\end{pmatrix}
\]

So, \( \text{Col} B = \{(b_1, b_2, b_3)^T | b_1 + 2b_2 - 2b_3 = 0\} \). Clearly, \( \text{Col} A = \text{Col} B \), denoted by \( V \). Then \( \{v_1, v_2\} \) and \( \{w_1, w_2\} \) are two bases for \( V \), which is a two-dimensional subspace of \( \mathbb{R}^3 \). \( \square \)

5. (15 points) Show that the hyperplane \( x + 2y + z - w = 0 \) is a subspace of \( \mathbb{R}^4 \). Find the dimension of and a basis for the hyperplane.

**Solution.** Denote the hyperplane by \( H \). Any vector in \( H \) is actually a solution to the homogeneous system

\[
x + 2y + z - w = 0.
\]

So we are to find a basis for the kernel of the coefficient matrix \( A = \begin{pmatrix} 1 & 2 & 1 & -1 \end{pmatrix} \), which is already in the echelon form. Clearly, \( y, z, w \) are free variables, and \( x = -2y - z + w \). So the general solution can be written as

\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
= \begin{pmatrix}
-2y - z + w \\
y \\
z \\
w
\end{pmatrix}
= \begin{pmatrix}
-2y \\
y \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
-z \\
0 \\
z \\
0
\end{pmatrix} + \begin{pmatrix}
w \\
w \\
0 \\
0
\end{pmatrix} = y \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} + z \begin{pmatrix}
-1 \\
0 \\
1 \\
0
\end{pmatrix} + w \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}.
\]
Therefore, $H = \text{span}\{(-2,1,0,0)^T, (-1,0,1,0)^T, (1,0,0,1)^T\}$. Since any span is a subspace, $H$ is a subspace of $\mathbb{R}^4$. Moreover, the three vectors spanning $H$ form a basis for $H$, and thus $\dim H = 3$. \qed

6. (18 points) Write the quadratic form $q(x) = 3x_1^2 + 5x_2^2 + 6x_3^2 + 4x_4^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_4 - x_3x_4$ in the form $q(x) = x^TKx$ for some symmetric matrix $K$. Determine if $q(x)$ is positive definite.

**Solution.** Matrix $K$ can be directly written out from the coefficients of the quadratic form $q$:

$$K = \begin{pmatrix}
3 & 2 & -1 & 0 \\
2 & 5 & 0 & -1 \\
-1 & 0 & 6 & -\frac{1}{2} \\
0 & -1 & -\frac{1}{2} & 4
\end{pmatrix}$$

We apply Gaussian to $K$:

$$\begin{pmatrix}
3 & 2 & -1 & 0 \\
2 & 5 & 0 & -1 \\
-1 & 0 & 6 & -\frac{1}{2} \\
0 & -1 & -\frac{1}{2} & 4
\end{pmatrix} \xrightarrow{R_2 - \frac{4}{3}R_1} \begin{pmatrix}
3 & 2 & -1 & 0 \\
0 & \frac{11}{3} & \frac{2}{3} & -1 \\
0 & \frac{20}{3} & \frac{20}{3} & -\frac{1}{2} \\
0 & -1 & -\frac{1}{2} & 4
\end{pmatrix} \xrightarrow{R_3 + \frac{4}{11}R_1} \begin{pmatrix}
3 & 2 & -1 & 0 \\
0 & \frac{11}{3} & \frac{2}{3} & -1 \\
0 & 0 & \frac{72}{11} & -\frac{7}{11} \\
0 & 0 & 0 & \frac{1069}{288}
\end{pmatrix}$$

Hence, $K$ is regular and has all positive pivots. Thus $K$ is positive definite, so is $q(x)$. \qed

**Bonus.** This problem may be much more challenging than the others. Allocate your time wisely.

(7 points) Prove that $(a + 2b + 3c)^2 \leq 6(a^2 + 2b^2 + 3c^2)$ for any real numbers $a, b, c$.

**Proof.** Let $w_1 = (1,1,1)^T$, $w_2 = (a,b,c)^T$. Using weighted inner product $\langle u, v \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$ on $\mathbb{R}^3$, we get

$$\langle w_1, w_2 \rangle = a + 2b + 3c, \quad \|w_1\|^2 = 6, \quad \|w_2\|^2 = a^2 + 2b^2 + 3c^2.$$ 

Apply Cauchy-Schwarz inequality, we obtain:

$$(a + 2b + 3c)^2 \leq 6(a^2 + 2b^2 + 3c^2).$$

(3 points) Prove the same inequality in a different way.

**Proof.** We can also prove the inequality by completing squares. Since

$$6(a^2 + 2b^2 + 3c^2) - (a + 2b + 3c)^2 = 5a^2 + 8b^2 + 9c^2 - 4ab - 12bc - 6ca = 2(a - b)^2 + 6(b - c)^2 + 3(c - a)^2 \geq 0,$$

we may obtain $(a + 2b + 3c)^2 \leq 6(a^2 + 2b^2 + 3c^2)$. Moreover, it’s clear that the equality holds if only and only if $a = b = c$. \qed