XI. Matrix Exponential

In this section we consider homogeneous linear systems with constant coefficients, i.e. autonomous versions of (LH). It is convenient to allow the coefficients and solutions to be complex-valued.

Indeed, complex-valued solutions are often helpful for constructing real-valued solutions to systems with real coefficients. Let \( A \in \mathbb{C}^{n \times n} \) be given and consider the system

\[
(ALH) \quad \dot{x}(t) = Ax(t).
\]

By a solution of (ALH) we mean a differentiable function \( x : \mathbb{R} \to \mathbb{C}^n \) such that (ALH) holds for all \( t \in \mathbb{R} \). By a matrix-valued solution of (ALH) we mean a differentiable function \( X : \mathbb{R} \to \mathbb{C}^{n \times n} \) such that \( \dot{X}(t) = AX(t) \) for all \( t \in \mathbb{R} \). Notice that a function \( X : \mathbb{R} \to \mathbb{C}^{n \times n} \) is a matrix-valued solution of (ALH) if and only if each column is a solution of (ALH). Notice also that if \( X \) is a matrix-valued solution of (ALH) and \( \xi \in \mathbb{C}^n \), \( C \in \mathbb{C}^{n \times n} \) then \( t \to X(t)\xi \) is a solution of (ALH) and \( t \to X(t)C \) is a matrix-valued solution of (ALH). It is straightforward to verify that a matrix-valued solution of (ALH) is invertible for all times if and only if it is invertible at 0.

Definition 11.1 For each \( t \in \mathbb{R} \) we define \( e^{tA} \in \mathbb{C}^{n \times n} \) to be the value at \( t \) of the matrix-valued solution \( X \) of (ALH) satisfying \( X(0) = I \), where \( I \) is the \( n \times n \) identity matrix.

Proposition 11.2 Let \( A, B \in \mathbb{C}^{n \times n} \) be given. Then

- (i) \( e^{0A} = I \);
- (ii) \( e^{(t+s)A} = e^{tA}e^{sA} \) for all \( s, t \in \mathbb{R} \);
- (iii) \( (e^{tA})^{-1} = e^{-tA} \) for all \( t \in \mathbb{R} \);
- (iv) \( Ae^{tA} = e^{tA}A \) for all \( t \in \mathbb{R} \);
- (v) \( e^{tA} = \sum_{m=0}^{\infty} \frac{(tA)^m}{m!} \) for all \( t \in \mathbb{R} \);
- (vi) If \( B \) is invertible then \( B^{-1}e^{tA}B = e^{tB^{-1}AB} \) for all \( t \in \mathbb{R} \);
- (vii) If \( A = \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_n) \) then \( e^{tA} = \text{diag} (e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}) \) for all \( t \in \mathbb{R} \).
(viii) \( \det(e^{tA}) = \exp[tr(A)t] \) for all \( t \in \mathbb{R} \);

(ix) \( Be^{tA} = e^{tAB} \) for all \( t \in \mathbb{R} \) if and only if \( AB = BA \);

(x) \( e^{(A+B)t} = e^{tA}e^{tB} \) for all \( t \in \mathbb{R} \) if and only if \( AB = BA \).