XII. Some Remarks on Eigenvectors and Generalized Eigenvectors

Let \( A \in \mathbb{C}^{n \times n} \) be given. A complex number \( \lambda \) is called an eigenvalue of \( A \) if the null space of \( \lambda I - A \) is nontrivial, i.e. if \( \mathcal{N}(\lambda I - A) \neq \{0\} \). Here \( I \) is the \( n \times n \) identity matrix, and for each \( B \in \mathbb{C}^{n \times n} \), \( \mathcal{N}(B) = \{ \xi \in \mathbb{C}^n : B\xi = 0 \} \). If \( \lambda \) is an eigenvalue of \( A \) the nonzero elements of \( \mathcal{N}(\lambda I - A) \) are called eigenvectors associated with \( \lambda \). The set of all eigenvalues of \( A \) is called the spectrum of \( A \) and is denoted by \( \sigma(A) \). The eigenvalues of \( A \) are precisely the roots of the characteristic equation.

\[
(12.1) \quad P_A(\lambda) = 0,
\]

where \( P_A : \mathbb{C} \to \mathbb{C} \) is the characteristic polynomial and is defined by

\[
(12.2) \quad P_A(\lambda) = \det(\lambda I - A) \quad \text{for all } \lambda \in \mathbb{C}.
\]

\( P_A \) is a polynomial of degree \( n \) and consequently \( \sigma(A) \) is nonempty and contains at most \( n \) elements. The algebraic multiplicity of an eigenvalue \( \lambda \) of \( A \) is defined to be its multiplicity as a root of (12.1) and is denoted by \( m_A(\lambda) \).

**Proposition 12.1:**

(i) \( tr(A) = \sum_{\lambda \in \sigma(A)} m_A(\lambda) \lambda \)

(ii) \( \det(A) = \prod_{\lambda \in \sigma(A)} \lambda^{m_A(\lambda)} \)

Notice that if \( \lambda \) is an eigenvalue of \( A \) and \( \xi \) is an associated eigenvector then \( e^{tA} \xi = e^{\lambda t} \xi \) for all \( t \in \mathbb{R} \). Consequently, if \( A \) has \( n \) linearly independent eigenvectors then we have a simple representation for \( e^{tA} \).

**Proposition 12.2:** Assume that \( \sigma(A) \) contains exactly \( n \) elements (i.e. that \( m_A(\lambda) = 1 \) for every \( \lambda \in \sigma(A) \)). Then \( \dim(\lambda I - A) = 1 \) for every \( \lambda \in \sigma(A) \) and there is a basis for \( \mathbb{C}^n \) consisting solely of eigenvectors of \( A \).

If \( \sigma(A) \) contains strictly less than \( n \) elements there may or may not be \( n \) linearly independent eigenvectors. However, there is always a basis that can be used to obtain a convenient representation for \( e^{tA} \).

**Definition 12.3:** Let \( \lambda \) be an eigenvalue of \( A \). A nonzero vector \( \xi \in \mathbb{C}^n \) is called a generalized eigenvector associated with \( \lambda \) if there is a positive integer \( k \) such that \( \xi \in \mathcal{N}((\lambda I - A)^k) \).
**Remark 12.3:** Let \( \lambda \) be an eigenvalue of \( A \) and \( \xi \) be an associated generalized eigenvector, and choose a positive integer \( k \) such that \((\lambda I - A)^k \xi = 0\). Notice that \((\lambda I - A)^m \xi = 0\) for all integers \( m \geq k \). Therefore, we have

\[
e^{tA} \xi = e^{t\lambda I} e^{t(A - \lambda I)} \xi = e^{t\lambda} e^{t(A - \lambda I)} \xi
\]

\[
= e^{t\lambda} \left( \sum_{m=0}^{\infty} \frac{t^m}{m!} (A - \lambda I)^m \right) \xi
\]

\[
= e^{t\lambda} \left( \xi + t(A - \lambda I)\xi + \ldots + \frac{t^{k-1}}{(k-1)!}(A - \lambda I)^{k-1}\xi \right)
\]

**Theorem 12.4:** For each \( \lambda \in \sigma(A) \) there is exactly one integer \( r_A(\lambda) \) satisfying

(i) \( 1 \leq r_A(\lambda) \leq m_A(\lambda) \)

(ii) \( \dim \mathcal{N}((\lambda I - A)^{r_A(\lambda)}) = m_A(\lambda) \)

(iii) \( \mathcal{N}((\lambda I - A)^m) = \mathcal{N}((\lambda I - A)^{r_A(\lambda)}) \) for all \( m \in \mathbb{N} \) with \( m \geq r_A(\lambda) \)

(iv) \( \mathcal{N}((\lambda I - A)^{r_A(\lambda)-1}) \neq \mathcal{N}((\lambda I - A)^{r_A(\lambda)}) \)

**Theorem 12.5:** There is a basis \( \mathcal{B} \) for \( \mathbb{C}^n \) with the following properties.

(i) Every element of \( \mathcal{B} \) is a generalized eigenvector of \( A \).

(ii) For every \( \lambda \in \sigma(A) \) there are exactly \( m_A(\lambda) \) elements of \( \mathcal{B} \) that belong to \( \mathcal{N}((\lambda I - A)^{r_A(\lambda)}) \).