1. Let $F = \mathbb{R}$ and $V = \mathbb{R}^4$. Determine whether or not the list of vectors is linearly independent.

   (a) $<1, 1, 2, 1>$, $<1, 1, 2, 2>$
   (b) $<1, -1, 1, 0>$, $<1, 2, 3, 4>$, $<1, 2, 3, 1>$, $<-1, 1, -1, 1>$, $<1, 0, 1, 0>$
   (c) $<-1, 1, 1, 1>$, $<0, 0, 0, 0>$, $<1, 3, 2, 1>$
   (d) $<1, 2, -1, 0>$, $<2, 1, 0, 1>$, $<4, 5, -2, 4>$

2. Let $F = \mathbb{R}$ and $V = \mathbb{R}^5$. Find a basis for the solution set of the equation $x_1 + 2x_2 - x_3 + x_4 - 2x_5 = 0$.

3. Prove or Disprove: Let $V$ be a finitely generated vector space and let $S, T, R$ be subspaces of $V$. Then $S + (T \cap R) = (S + T) \cap (S + R)$.

4. Let $F = \mathbb{R}$ and let $n \in \mathbb{Z}^+$ be given. Let $P_n(\mathbb{R})$ denote the vector space of all real polynomials of the degree $\leq n$. Assume that $f_0, f_1, \ldots, f_n \in P_n(\mathbb{R})$ satisfy $f_0(\pi) = f_1(\pi) = f_2(\pi) = \ldots = f_n(\pi) = 0$. Show that the list $f_0, f_1, f_2, \ldots, f_n$ is linearly dependent.

5. Let $F$ be a field and $V, W$ be vector spaces over $F$. Let $L : V \rightarrow W$ be a mapping such that $L(u + v) = L(u) + L(v)$ and $L(\lambda u) = \lambda L(u)$ for all $u, v \in V$, $\lambda \in F$. Let

   $$T = \{ L(u) : u \in V \}.$$ 

   Show that $T$ is a subspace of $W$.

6. Let $F = \mathbb{R}$ and let $P_3(\mathbb{R})$ be the vector space of all real polynomials of degree $\leq 3$. Let $S = \{ f \in P_3(\mathbb{R}) : f(1) = 2 \int_0^1 f(x)dx \}$.

   Show that $S$ is a subspace of $P_3(\mathbb{R})$ and find a basis for $S$.

7. Let $F = \mathbb{R}$ and $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 5 & 2 & 1 \\ -2 & 5 & 5 & 4 \end{pmatrix}$.

   Find a $3 \times 2$ matrix $A'$ such that $A'$ is row equivalent to $A$, all zero rows of $A'$ are below the nonzero rows, and the nonzero rows of $A'$ are in echelon form.
8. Let $V$ be an eight dimensional vector space and let $S, T$ be subspaces of $V$ with $\dim(S) = 5$ and $\dim(T) = 6$. What is the smallest possible dimension of $S \cap T$?

9. Prove or Disprove: Let $S, T, R$ be subspaces of a finitely generated vector space $V$. If $S + T = S + R$ then $T = R$.

10. In this problem, we will write complex numbers in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$; for such a number we write

$$Re(a + bi) = a, \quad Im(a + bi) = b.$$ 

Let $V = \mathbb{C}^4$.

(a) Assume that $\mathbb{F} = \mathbb{R}$. Which of the following are subspaces?

i. $\{ < z_1, z_2, z_3, z_4 > : z_1 + z_4 = 0 \}$

ii. $\{ < z_1, z_2, z_3, z_4 > : z_1 + z_4 = 1 \}$

iii. $\{ < z_1, z_2, z_3, z_4 > : z_1 + z_4 = z_2 \}$

iv. $\{ < z_1, z_2, z_3, z_4 : Re(z_1) = 0 \}$

v. $\{ < z_1, z_2, z_3, z_4 : Im(z_1) = 0 \}$

vi. $\{ < z_1, z_2, z_3, z_4 > : z_1z_2 = 0 \}$

(b) How would the answers to part (a) change if the field were changed from $\mathbb{R}$ to $\mathbb{C}$.

11. Let $\mathbb{F} = \mathbb{R}$ and $V = \mathcal{F}(\mathbb{R})$ the set of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$. Let $f_1(x) = e^x, \ f_2(x) = e^{2x}, \ f_3(x) = e^{3x}$. Determine whether or not $f_1, f_2, f_3$ are linearly independent.