V. Periodic Systems

Let \( T > 0 \) and \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be given. Throughout this section we assume that \( f \) is continuous, has the uniqueness property, and satisfies

\[
(5.1) \quad f(t + T, z) = f(t, z) \quad \forall t \in \mathbb{R}, \ z \in \mathbb{R}^n.
\]

By a \textit{T-periodic solution} of

(5.2) \quad x(t + T) = x(t) \quad \forall t \in \mathbb{R}.

The following lemma is a direct consequence of the uniqueness property and (5.1).

\textbf{Lemma 5.1:} Let \( x \) be a noncontinuable solution of (DE) and let \( t_0 \in \text{Dom}(x) \) be given. If \( t_0 + T \in \text{Dom}(x) \) and \( x(t_0 + T) = x(t_0) \) then \( x \) is a \( T \)-periodic solution.

By using Lemma 5.1, together with Theorem 4.11 and Brouwer’s fixed point Theorem, we obtain the following important result.

\textbf{Theorem 5.2:} Let \( S \) be a nonempty, closed, bounded, convex subset of \( \mathbb{R}^n \) and let \( t_0 \in \mathbb{R} \) be given. Assume that for every \( x_0 \in S \) the unique noncontinuable solution \( x \) of

(IVP) \quad \dot{x}(t) = f(t, x(t)) ; \ x(t_0) = x_0

satisfies \( t_0 + T \in \text{Dom}(x) \) and \( x(t_0 + T) \in S \). Then (DE) has a \( T \)-periodic solution.

In order to apply Theorem 5.2 in practice, the key step is to find a suitable set \( S \). The following lemma, which is a consequence of the Mean Value Theorem, is often helpful for this purpose.

\textbf{Lemma 5.3:} Let \( I \subset \mathbb{R} \) be an open interval and let \( t_0 \in I \) and \( \alpha, \alpha', \beta, \beta' \in \mathbb{R} \) with \( \alpha' < \alpha \) and \( \beta < \beta' \) be given.
(a) If \( F(t_0) \geq \alpha \) and \( \dot{F}(s) \geq 0 \) for all \( s \in I \cap [t_0, \infty) \) such that \( \alpha' \leq F(s) \leq \alpha \) then \( F(t) \geq \alpha \) for all \( t \in I \cap [t_0, \infty) \).

(b) If \( F(t_0) \leq \beta \) and \( \dot{F}(s) \leq 0 \) for all \( s \in I \cap [t_0, \infty) \) such that \( \beta \leq F(s) \leq \beta' \) then \( F(t) \leq \beta \) for all \( t \in I \cap [t_0, \infty) \).

**Theorem 5.4:** Let \( \Gamma_1 \geq 0 \) and \( \Gamma_2 > 0 \) be given. Assume that

\[
(5.3) \quad z \cdot f(t, z) \leq 0 \quad \text{for all } t \in \mathbb{R}, \; z \in \mathbb{R}^n \text{ with } \Gamma_1 \leq ||z||_2 \leq \Gamma_2.
\]

Then \( (DE) \) has a \( T \)-periodic solution.