Let \( n \in \mathbb{Z}^+ \) be given. Let \( \mathbb{F} \) be a field and \( V \) be a vector space over \( \mathbb{F} \) with \( \dim V = n \). Let \( v_1, v_2, \ldots, v_n \) be a basis for \( V \). We denote by \( \mathbb{F}^{n \times 1} \) the set of all \( n \times 1 \) matrices with entries from \( \mathbb{F} \). Consider the linear mapping \( C : V \to \mathbb{F}^{n \times 1} \) defined by\
\[ C v_i = e^t_i \quad i = 1, 2, \ldots, n \]
where \( e^t_i \) is the \( n \times 1 \) matrix whose \( i \)th entry (row) is 1 and all other entries are 0. Notice that if \( v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n \) then\
\[ Cv = \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{array} \right) \]
We call \( C \) the component mapping for the basis \( v_1, v_2, \ldots, v_n \).

I. Matrix for \( T \in L(V, V) \):

Let \( T \in L(V, V) \) be given. We want to find an \( n \times n \) matrix \( A \) such that\
\[ C(Tv) = A(Cv) \quad \text{for all } v \in V. \]
Notice that if we have such a matrix, then we can compute \( Tv \) for a given \( v \in V \) as follows: Choose \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F} \) such that\
\[ v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n \]
and let\
\[ \left( \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_n \end{array} \right) = A \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{array} \right). \]
Then we have

\[ T v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n. \]

Let \( x \in \mathbb{F}^{n \times 1} \) be given. If we put \( v = C^{-1} x \) in (3) we obtain

\[ Ax = C(TC^{-1} x) \]

In particular, we have

\[ Ae_j^t = C(TC^{-1} e_j^t) = C(Tv_j) \quad j = 1, 2, \ldots, n. \]

Notice that \( Ae_j^t \) is simply the \( j \)th column of \( A \). Therefore, the \( j \)th column of \( A \) simply consists of the coefficients needed to express \( Tv_j \) as a linear combination of \( v_1, v_2, \ldots, v_n \).

**Change of Basis for \( T \in L(V, V) \):**

Let \( T \in L(V, V) \) be given and let \( A \) be the matrix for \( T \) relative to the basis \( v_1, v_2, \ldots, v_n \). Let \( w_1, w_2, \ldots, w_n \) be a second basis for \( V \) satisfying

\[ w_i = \sum_{j=1}^{n} \mu_{ji} v_j, \quad i = 1, 2, \ldots, n. \]

Let \( B \) be the matrix for \( T \) relative to \( w_1, w_2, \ldots, w_n \). We want to find the relationship between \( A \) and \( B \).

Let \( S \) be the \( n \times n \) matrix whose \( ij \) entry is \( \mu_{ij} \). Then, by (9), \( S \) is the matrix relative to \( v_1, v_2, \ldots, v_n \) of the linear transformation \( S \in L(V, V) \) characterized by

\[ Sv_i = w_i, \quad i = 1, 2, \ldots, n. \]

Let \( D \) be the component mapping for \( w_1, w_2, \ldots, w_n \) and observe that

\[ Bx = DTD^{-1} x \quad \text{for all } x \in \mathbb{F}^{n \times 1}. \]
by virtue of Part I. It follows from (10) that

(12) \[ DSv = Cv \quad \text{for all } v \in V \]

which yields

(13) \[ Sv = D^{-1}Cv \quad \text{for all } v \in V. \]

Using the results of Part I, we find that

\[
Sx = CD^{-1}CC^{-1}x \\
= CD^{-1}x.
\]

Moreover, by (3), we have

(15) \[ Tv = C^{-1}ACv \quad \forall v \in V. \]

Substitution of (15) into (11) yields

(16) \[ Bx = D(C^{-1}AC)D^{-1}x \\
= S^{-1}ASx \]

by virtue of (14).

III. Matrix for \( T \in L(V, W) \):

Let \( m \in \mathbb{Z}^+ \) be given and let \( W \) be a vector space over \( \mathbb{F} \) with \( \dim W = m \). Let \( u_1, u_2, \ldots, u_m \) be a basis for \( W \) and let \( E \in L(W, \mathbb{F}^{m \times 1}) \) be the component mapping for \( u_1, u_2, \ldots, u_m \).

Let \( T \in L(V, W) \) be given. We want to find an \( m \times n \) matrix \( A \in \mathbb{F}^{m \times n} \) such that

(17) \[ ETv = ACv \quad \forall v \in V. \]
Let $x \in \mathbb{F}^{n \times 1}$ be given. If we put $v = C^{-1}x$ in (17) we get

(18) \hspace{1cm} ET C^{-1}x = Ax \hspace{.5cm} \forall x \in \mathbb{F}^{n \times 1}.

To understand what the matrix $A$ looks like, we set $x = e^t_j$ in (18) to get

(19) \hspace{1cm} ET C^{-1}e^t_j = Ae^t_j \hspace{1cm} ET v_j = Ae^t_j

which says that the $j$th column of $A$ consists of the coefficients required to express $Tv_j$ as a linear combination of $u_1, u_2, \ldots u_m$. We call $A$ the matrix for $T$ relative to the bases $v_1, v_2, \ldots v_n$ and $u_1, u_2, \ldots u_m$.

IV. Inner Product Spaces:

Suppose that $\mathbb{F} = \mathbb{R}$, $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is an inner product, and that $u_1, u_2, \ldots u_n$ is an orthonormal basis for $V$. Since

(20) \hspace{1cm} v = \sum_{i=1}^{n} (v, u_i)u_i \hspace{.5cm} \forall v \in V,

it follows that

(21) \hspace{1cm} Cv = \begin{pmatrix} (v, u_1) \\ (v, u_2) \\ \vdots \\ (v, u_n) \end{pmatrix}

Let $T \in L(V, V)$ be given. It follows from (8) and (21) that if $A$ is the matrix for $T$ relative to $u_1, u_2, \ldots u_n$ then

(22) \hspace{1cm} A_{ij} = (Tu_j, u_i),

where $A_{ij}$ is the entry of $A$ from row $i$ and column $j$. 