II. Preliminaries

Let $n$ be a positive integer. We denote by $\mathbb{R}^n$ the set of all $n$-tuples of real numbers $x = (x_1, x_2, \ldots, x_n)$ with the usual notions of addition and scalar multiplication. We use the same symbol $0$ to denote the real number zero as well as the zero element of $\mathbb{R}^n$ when there is no danger of confusion.

By a norm on $\mathbb{R}^n$ we mean a function $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ satisfying

\begin{enumerate}
  \item $\|x\| > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$,
  \item $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$,
  \item $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$.
\end{enumerate}

Property (iii) is called the \textit{triangle inequality}. An important consequence of this property is that if $a$ and $b$ are real numbers with $a < b$ and $g : [a, b] \to \mathbb{R}^n$ is continuous then

\begin{equation}
\| \int_a^b g(t) dt \| \leq \int_a^b \|g(t)\| dt.
\end{equation}

All norms on $\mathbb{R}^n$ are \textit{equivalent} in the sense that if $\| \cdot \|$ and $\| \| \| \|$ are norms then there exist constants $m, M > 0$ such that

\begin{equation}
m\|x\| \leq \|\|x\|| \leq M\|x\| \quad \text{for all } x \in \mathbb{R}^n.
\end{equation}

For each $p \in [1, \infty)$ the function $\| \cdot \|_p : \mathbb{R}^n \to \mathbb{R}$ defined by

\begin{equation}
\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \quad \text{for all } x \in \mathbb{R}^n
\end{equation}

is a norm. In addition, the function $\| \cdot \|_\infty : \mathbb{R}^n \to \mathbb{R}$ defined by

\begin{equation}
\|x\|_\infty = \max \{|x_i| : i = 1, 2 \ldots n\}
\end{equation}

is also a norm. Observe that
(2.5) \[ \|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty \quad \text{for all } x \in \mathbb{R}^n. \]

The case \( p = 2 \) is especially important because \( \| \cdot \|_2 \) is associated with an inner product. Recall that the dot product or inner product of \( x, y \in \mathbb{R}^n \) is defined by

(2.6) \[ x \cdot y = \sum_{i=1}^{n} x_i y_i, \]

so that

(2.7) \[ \|x\|_2 = \sqrt{x \cdot x} \quad \text{for all } x \in \mathbb{R}^n. \]

The Cauchy-Schwarz inequality, which says that

(2.8) \[ |x \cdot y| \leq \|x\|_2 \|y\|_2 \quad \text{for all } x, y \in \mathbb{R}^n, \]

will play an important role in our analysis of differential equations.

The norm \( \| \cdot \|_2 \) is called the Euclidean norm. An especially useful feature of this norm is that if \( I \) is an interval, \( g : I \rightarrow \mathbb{R}^n \) is differentiable then the function \( t \mapsto \|g(t)\|_2^2 \) is differentiable on \( I \) and

(2.9) \[ \frac{d}{dt} (\|g(t)\|_2^2) = 2g(t) \cdot \dot{g}(t) \quad \text{for all } t \in I. \]

For each \( \delta > 0 \) and \( x \in \mathbb{R}^n \), we put

(2.10) \[ B_\delta(x) = \{ y \in \mathbb{R}^n : \|y - x\|_2 < \delta \}. \]

Let \( D \) be a subset of \( \mathbb{R}^n \). A point \( x_0 \in D \) is said to be an interior point of \( D \) if there exists \( \delta > 0 \) such that \( B_\delta(x_0) \subset D \). The set of all interior points of \( D \) is called the interior of \( D \) and is denoted by \( \text{int}(D) \). We say that \( D \) is open if \( \text{int}(D) = D \). We say that \( D \) is closed if \( \mathbb{R}^n \setminus D \) is open.

A point \( x_0 \in \mathbb{R}^n \) is called a boundary point of \( D \) if

(2.11) \[ \forall \delta > 0, \ B_\delta(x_0) \cap D \neq \emptyset \quad \text{and} \quad B_\delta(x_0) \cap (\mathbb{R}^n \setminus D) \neq \emptyset, \]

i.e. for every \( \delta > 0 \), \( B_\delta(x_0) \) contains points that belong to \( D \) as well as points that do not belong to \( D \). The set of all boundary points of \( D \) is called the boundary of \( D \) and is denoted by \( \partial D \). It is not too difficult to show that \( D \) is closed if and only if \( \partial D \subset D \). We say that \( D \) is bounded if there exists \( M \in \mathbb{R} \) such that
(2.12) \[ \|x\|_2 \leq M \text{ for all } x \in D. \]

**Remark 2.1:** In view of the equivalence of norms on \( \mathbb{R}^n \), the notions of interior, boundary, open set, closed set, bounded set do not change if \( \| \cdot \|_2 \) is replaced by any other norm in (2.10).

We say that \( D \) is **convex** if

(2.13) \[ tx + (1 - t)y \in D \quad \text{for all } x, y \in D, \ t \in [0, 1], \]

i.e., \( D \) contains the line segment joining each pair of points in \( D \). The following result will be very useful.

**Brouwer’s Fixed-Point Theorem:** Let \( D \) be a nonempty, closed, bounded, convex subset of \( \mathbb{R}^n \) and assume that \( f : D \to \mathbb{R}^n \) is continuous. If \( f(x) \in D \) for every \( x \in D \) then there is at least one point \( x^* \in D \) such that \( f(x^*) = x^* \).

Let \( m \) be a positive integer. Then \( \mathbb{R}^m \times \mathbb{R}^n \) can be identified with \( \mathbb{R}^{m+n} \).

**Remark 2.2:** Let \( S \) be a subset of \( \mathbb{R}^m \) and \( T \) be a subset of \( \mathbb{R}^n \).

(i) If both \( S \) and \( T \) are open, then \( S \times T \) is open.

(ii) If both \( S \) and \( T \) are closed, then \( S \times T \) is closed.

(iii) If both \( S \) and \( T \) are bounded, then \( S \times T \) is bounded.

(iv) If both \( S \) and \( T \) are convex, then \( S \times T \) is convex.