Here a proof will be presented (due to Tait) of strong normalization of the typed lambda calculus with all the propositional types (sum, product, function, and bottom). There may be some errors. Let me know if you find any!

First we begin with defining the types:

**Definition 1.** Fix a set $A$ which will serve as our set of atomic types, which includes a special symbol $\bot$. Then, we define, inductively, the set $T$ of types which is the smallest set containing $A$ and closed under the following:

- **(Arrow/Implication)** If $\alpha, \beta \in T$ then $\alpha \to \beta \in T$
- **(Product/Conjunction)** If $\alpha, \beta \in T$ then $\alpha \times \beta \in T$
- **(Sum/Disjunction)** If $\alpha, \beta \in T$ then $\alpha + \beta \in T$

And the following typing rules (presented in a natural deduction system)

\[
\begin{align*}
\Gamma, x : \alpha & \vdash x : \alpha \\
\Gamma & \vdash M : \alpha \\
\Gamma & \vdash N : \beta \\
\Gamma & \vdash \langle M, N \rangle : \alpha \times \beta \\
\Gamma, x : \alpha & \vdash M : \beta \\
\Gamma & \vdash (\lambda x. M) : \alpha \to \beta \\
\Gamma & \vdash \pi_1 M : \alpha_i \\
\Gamma & \vdash \pi_2 M : \alpha_2 \\
\Gamma & \vdash \text{case}(\text{in}_i(M); x_1.N_1; x_2.N_2) : \beta \\
\Gamma & \vdash c_\alpha M : \alpha
\end{align*}
\]

This gives the rules for typings. This suffices for discussing provability. The reduction rules however are interesting in they give us the power to “simplify” proofs. The following is our reduction rules:

\[
\begin{align*}
(\lambda x. M)N & \mapsto M[x := N] \\
\pi_1\langle M_1,M_2 \rangle & \mapsto M_i \\
\text{case}(\text{in}_i(M); x_1.N_1; x_2.N_2) & \mapsto N_i[x_i := M]
\end{align*}
\]

And also reduction rules to make these congruences (eg. If $M \mapsto M'$ then $(\lambda x. M) \mapsto \lambda x. M'$).

The following theorem implies the cut elimination theorem for propositional logic via the Curry Howard isomorphism. Here, reductions of the terms correspond to reductions of the proof. A normal form corresponds to a normal proof in the natural deduction system, which gives a cut-free proof in the sequent calculus.

**Theorem 1 (Strong Normalization).** In the above system, if $M$ is a term, then any sequence of reductions terminates. That is, there is no infinite reduction paths.

**Proof.** We do this proof in two steps. In the first step we define a (ostensibly) smaller subset of the typed lambda terms and show that it has nice closure properties and is strongly normalizing.

In the second step, we show that all terms are in this subset.

For every type $\alpha$ we define the set of strongly computable terms of type $\alpha$, which we denote $\text{SC}_\alpha$ inductively:

- If $\alpha$ is atomic, then $\text{SC}_\alpha = \{ M \text{ of type } \alpha \mid M \text{ is strongly normalizing } \}$
- If $\alpha = \beta \to \gamma$ then $\text{SC}_{\beta \to \gamma} = \{ M \text{ of type } \beta \to \gamma \mid \forall N \in \text{SC}_\beta . MN \in \text{SC}_\gamma \}$
- If $\alpha = \beta \times \gamma$ then $\text{SC}_{\beta \times \gamma} = \{ M \text{ of type } \beta \times \gamma \mid (\pi_1 M) \in \text{SC}_\beta \land (\pi_2 M) \in \text{SC}_\gamma \}$
Lemma 1. If $M \in SC_{\alpha}$ for some $\alpha$ then $M$ is strongly normalizing.

Proof. Do induction on type.

In the case when the type is atomic, it is easy to see.

If $M \in SC_{\alpha \rightarrow \beta}$ then for every $N \in SC_{\alpha}$ we have $MN \in SC_{\beta}$. By induction hypothesis, $MN$ is strongly normalizing. Thus, if there was an infinite path in $M$, we could build one in $MN$.

If $M \in SC_{\alpha \times \beta}$ then $\pi_1(M) \in SC_{\alpha}$. This is strongly normalizing by induction hypothesis. Thus $M$ is strongly normalizing as otherwise we could build an infinite path in $\pi_1(M)$.

If $M \in SC_{\alpha + \beta}$ then case $M; x : \beta \rightarrow x; y : \gamma, y) \in SC_{\beta} \cup SC_{\gamma}$. The sets are disjoint (terms have unique types), as it’s either in one or the other. Without loss on generality, it is in $SC_{\beta}$. By induction hypothesis, it is strongly normalizing. Thus $M$ is strongly normalizing as otherwise we could build an infinite reduction path in case $M; x : \beta, x; y : \gamma, y) \in SC_{\beta} \cup SC_{\gamma}$.

Lemma 2. If $M \rightarrow M'$ and $M \in SC_{\beta}$ then $M' \in SC_{\beta}$

Proof. As $M \in SC_{\alpha}$ it has a longest reduction path. We denote this $h(M)$.

If $M$ is atomic, then $M \rightarrow M'$ and $h(M') < h(M)$ so $M'$ is strongly normalizing and thus in $SC_{\alpha}$.

If $M$ is of type $\beta \rightarrow \gamma$ suppose that $M \rightarrow M'$. Fix $N \in SC_{\beta}$. Want to show that $M'N \in SC_{\gamma}$. Well, $MN \in SC_{\alpha}$ by definition of $SC_{\alpha \rightarrow \beta}$. Thus by induction hypothesis, $MN \rightarrow M'N$ and so $M'N \in SC_{\gamma}$.

If $M$ is of type $\beta \times \gamma$ then suppose $M \rightarrow M'$. Then $\pi_1(M) \rightarrow \pi_1(M')$ and $\pi_2(M) \rightarrow \pi_2(M')$. By induction hypothesis, $\pi_1(M') \in SC_{\beta}$ and $\pi_2(M') \in SC_{\gamma}$. So $M' \in \beta \times \gamma$.

If $M$ is of type $\beta + \gamma$ then suppose $M \rightarrow M'$. Then case $M; x : \beta, x; y : \gamma, y) \in SC_{\beta} \cup SC_{\gamma}$. As they as disjoint, we have this in exactly one of $SC_{\beta}$ or $SC_{\gamma}$. Without loss of generality, case $M; x : \beta, x; y : \gamma, y) \in SC_{\beta}$. By induction hypothesis, case $M'; x : \beta, x; y : \gamma, y) \in SC_{\beta} \subseteq SC_{\beta} \cup SC_{\gamma}$, and so $M' \in SC_{\beta + \gamma}$.

We define a term to be neutral if they have one of the following forms:

- $x$, ie variables are neutral
- $\pi_c x$
- case $M; x_1.N_1; x_2.N_2$
- $c_{\alpha} N$
- $MN$

Lemma 3. If $M$ is neutral and for every one step reduction $M \rightarrow M'$ we have $M' \in SC_{\alpha}$ then $M \in SC_{\alpha}$

Proof. We do induction on the type.

In the case where $M$ is of type $\alpha$ for $\alpha$ atomic then it’s clear that if every one step reduction makes the term strongly normalizing then $M$ itself is strongly normalizing.

In the case where $M$ if of the type $\alpha \rightarrow \beta$ then take $N \in SC_{\alpha}$. We’d like to show that $MN \in SC_{\beta}$. $MN$ is itself neutral, so it suffices to check that every one step reduction of $MN$ is in $SC_{\beta}$. Note that by neutrality of $M$, $MN$ itself cannot be a reducible expression. Thus any one step reduction is either entirely contained in $M$ or $N$. $N$ is strongly normalizing so we do a secondary induction on the longest reduction path in $N$. In the first case, $MN \rightarrow M'N$ is in $SC_{\beta}$ by the assumption. In the second case, the longest reduction path of the term $N$ decreases, so it is done by the secondary induction hypothesis.

In the case where $M$ is of type $\alpha_1 \times \alpha_2$, note that $\pi_1 M$ and $\pi_2 M$ are neutral, so it suffices to just check the one step reductions in these (note that $\pi_1 M$ is not itself a redex as $M$ is neutral and thus not a (\()$) term. But by induction hypothesis, every one step reduction of $M$ lies in $SC_{\alpha_1 \times \beta}$, so every one step reduction of $\pi_1 M$ lies in $SC_{\alpha_1}$. Thus $\pi_1 M$ is itself in $SC_{\alpha_1}$ so $M$ is in $SC_{\alpha_1 \times \alpha_2}$ by definition of that set.

In the case where $M$ is of type $\alpha_1 + \alpha_2$ we have that case $M; x : \alpha_1, x; y : \alpha_2, y)$ is itself neutral, so it suffices to check all one step reductions here. They all must be in $M$ (M itself cannot be an in term as it is neutral). If it lies in $M$, by assumption, they all land in $SC_{\alpha_1} \cup SC_{\alpha_2}$. Thus case $M; x : \alpha_1, x; y : \alpha_2, y)$ is itself in $SC_{\alpha_1} \cup SC_{\alpha_2}$ induction hypothesis. Thus $M \in SC_{\alpha_1 + \alpha_2}$.
Now we are in the position to show every term is in SC\(_{\alpha}\)

We do this by induction on terms. We will have the stronger induction hypothesis: If \(M\) is a term of type \(\alpha\) then \(M[x := N]\) is in SC\(_{\alpha}\) when \(N \in \text{SC}_{\beta}\) (where all these types are appropriate).

If \(M\) is a variable \(x\), then this is immediate.

If \(M = PQ\) then we must have \(P : \alpha \to \beta\) and \(Q : \alpha\). By induction hypothesis, \(P \in \text{SC}_{\alpha \to \beta}\) and \(Q \in \text{SC}_{\alpha}\), so \(PQ \in \text{SC}_{\beta}\). The substitution is trivial.

If \(M = \lambda x. P\) then \(M : \alpha \to \beta\), where \(P : \beta\). Then take \(N \in \text{SC}_{\alpha}\). Want to show that \(MN \in \text{SC}_{\beta}\). Well, \(MN\) is neutral. Suffices to check all one step reductions, and do a secondary induction on \(h(P) + h(N)\). If the one step reduction is contained in \(P\) or \(N\) then we are done. Otherwise the reduction is the redex \(MN\). By this reduces to \(P[x := N]\) by the stronger induction hypothesis, as \(N \in \text{SC}_{\alpha}\) we have \(P[x := N] \in \text{SC}_{\beta}\). Note that checking the substitution condition here is trivial.

If \(M = \langle P, Q \rangle\) having type \(\alpha \times \beta\) by induction hypothesis, \(P \in \text{SC}_{\alpha}\) and \(Q \in \text{SC}_{\beta}\) so we are done. Checking substitution condition is trivial.

If \(M = \pi_i P\) where \(P\) is of type \(\alpha \times \beta\). By induction hypothesis, \(P\) is in \(\text{SC}_{\alpha \times \beta}\). Thus by definition of this set, we are done. Checking the substitution condition is trivial.

If \(M = \text{in}_i(P)\) has type \(\alpha_1 + \alpha_2\) we have by induction hypothesis, \(P \in \text{SC}_{\alpha_i}\). To check \(M \in \text{SC}_{\alpha_1 + \alpha_2}\) it is necessary to check case(\(\text{in}_i(P); x_1 : \alpha_1; x_2 : \alpha_2\)). This term is neutral. It suffices to check all one step reductions. Do a secondary induction on the longest path in \(P\). If \(P\) gets reduced, then it’s longest path gets reduced so we are done by the secondary hypothesis. Otherwise, the redex reduces the term itself. And it reduces to \(P\) itself, which is in \(\text{SC}_{\alpha_i}\), so we are done.

If \(M = \text{case}(P; x_1 : \alpha_1; N_1; x_2 : \alpha_2; N_2)\) of type \(\beta\), then observe this term is neutral, and by induction hypothesis \(N_1, N_2 \in \text{SC}_{\beta}\) and \(P \in \text{SC}_{\alpha_1 + \alpha_2}\). Do an secondary induction on the sum \(h(P) + h(N_1) + h(N_2)\). If the one step reduction is done in \(P\), \(N_1\), or \(N_2\) then we are done by the secondary hypothesis. Otherwise, it must be the redex itself. In this case, \(P\) must have the form \(\text{in}_i(P')\). Then the reduction done is \(N_1[x_1 := P']\), which is in \(\text{SC}_{\beta}\) by the stronger induction hypothesis.

If \(M = c\alpha N\), this is neutral and all the one step reductions live in \(N\), so we have the result.