1 The diagram $D(T)$ of a theory $T$

Pay special attention to theorem 1.21 (p. 187) (if the diagram of a countable theory is uncountable, then it is of size $2^\aleph_0$) ... hence the spectrum function must also have value continuum at $\aleph_0$. The proof of 1.21 is prototypical of many constructions of “many objects” in model theory. (I wasn’t aware till now that Łoś also proved this theorem. I thought it came from the Berkeley school (Tarski?). Another Łoś Theorem!...)

A few more remarks on the proof of 1.21. It is essentially the ‘perfect set argument’ due to Cantor, but adapted to our situation.

The statement is similar to the Vaught Conjecture (for countable $T$, if $I(\aleph_0, T) > \aleph_0$ then $I(\aleph_0, T) = 2^\aleph_0$). Robin Knight from Oxford announced in 2002 a counterexample to the Vaught Conjecture. I do not know the status of his proof at present.

Interestingly, for some model theorists the Vaught Conjecture seems to have been a major driving force. Shelah, however, dismisses the Vaught Conjecture as a secondary problem. His reasons may be connected to the fact that he is more interested in behaviour of model-theoretic functions “at many cardinals”, like the Main Gap. There are however very interesting results, such as the proof of the Vaught Conjecture for (up to) superstable theories.

2 Large models

Three notions of “largeness” for models are crucial: saturation, homogeneity and universality. We have already used some weak instances of saturation (in very specific ways) when we did the (Cantor style) back-and-forth proof of categoricity of $T_{ord}$. 

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For $T_{\text{ind}}$ similar ideas worked: for the successor step find a realization of a ‘challenge’ type. Use density (and no endpoints) for $T_{\text{ord}}$ and the definition in the case of $T_{\text{ind}}$. Notice that the construction carried so far was finite, so there are only finitely many ‘restrictions’. The Steinitz theorem had analogous situations in its back-and-forth proof: we used either algebraic closure or large size to find an element corresponding to the challenge in the back and forth successor step.

Saturation captures the theme common to those three proofs ($M$ is saturated iff all types over sets of parameters smaller than $M$ are realized in $M$). And indeed,

**Theorem 1** Let $M, N$ be $L$-structures, both saturated and of the same cardinality. Then $M \approx N$.

The proof is again a back and forth argument, that has to be done in a way slightly more careful than in the three specific cases mentioned.