Sixth Week

February 17, 2003

1 Amalgamation, Robinson and Craig

Among other things, we proved that abstract elementary classes satisfy amalgamation as a consequence of

\textbf{Theorem 1 (Abraham Robinson)} If $T_0 = T_1 \cap T_2$ are consistent theories, $T_1$ in $L_1$, $T_0$ is $L_0$-complete, and $L_0 = L_1 \cap L_2$, then the $L_1 \cup L_2$-theory $T_1 \cup T_2$ is consistent.

The proof was presented in class by Henrik Forssell. It is actually stronger than the amalgamation property. Compactness is used in the proof for the “strong joint embedding” property step.

Heuristically, the Robinson Consistency Lemma says that, as long as the “common part” of two theories is complete, there can be no contradiction arising from symbols that only exist on either one of the theories.

This intuition is somehow reinforced by the so-called Craig Interpolation Theorem:

\textbf{Theorem 2 (Craig)} Let $L_0 = L_1 \cap L_2$ be languages, and let $\varphi$ be a sentence in $L_1$ and $\psi$ a sentence in $L_2$ such that $\varphi \vdash \psi$. Then there exists a sentence $\theta$ in $L_0$ such that

$$\varphi \vdash \theta \vdash \psi$$

($\theta$ is called an “interpolant” sentence).

Heuristically, this goes along the same lines as the Robinson lemma. It says that, somehow, “only the common language” was needed in $\varphi$’s implication of $\psi$.

As an exercise (not necessarily to return to me), prove that Robinson implies Craig. Also (but slightly more difficult): Craig implies Robinson.
The Craig Interpolation Theorem is actually one of a large family of “Interpolation Theorems” that were much studied in the 70s by Shelah, Makowsky and others in the context of “abstract model theory”. There are quite natural topological proofs of Interpolation that adapt quite well to non first order cases.

As another exercise (that pays off if you want to understand some abstract cases), you may try to “rewrite” Craig’s Interpolation Theorem in topological terms: for fixed language $L$, let $X_L$ be the space whose points are equivalence classes mod $\equiv$ of $L$-structures, and basic open sets are sets of the form $U_\varphi = \{ \mathfrak{A} \in X_L | \mathfrak{A} \models \varphi \}$, for each $L$-sentence $\varphi$. Check what compactness means in this context. Notice that closed sets are determined by $L$-theories. Notice that $X_L$ is a totally disconnected Hausdorff space. Translate Craig to this context: notice it corresponds to a strong form of separation.