1 Spectra

There are several spectrum functions in model theory: among others,

existence spectrum,
categoricity spectrum,
saturation spectrum,
homogeneity spectrum,
stability spectrum.

Really, a complete study of all of those spectra take much longer than a one-semester course. We will study to a large extent the categoricity spectrum for first order logic. However, let us look at the function that computes, for a given (FO) theory and a given cardinal, the number of models of the theory in that cardinal.

\[ I(\lambda, T) := |\{ M/ \models T, |M| = \lambda \}|. \]

Easy: \[ 0 \leq I(\lambda, T) \leq 2^{\lambda+N_0+|T|}. \] So, once we get “past the noise”, the upper bound is essentially \( 2^\lambda \).

1. \( I(\aleph_0, T_{ACF_0}) = \aleph_0, I(\lambda, T_{ACF_0}) = 1 (\lambda \text{ uncountable}). \)

2. \( I(\aleph_0, ZFC) = 2^{\aleph_0} = I(\aleph_0, PA). \) Same maximal number of models at any cardinality.
Exercise: write upward and downward Löwenheim-Skolem as theorems on $I(\lambda, T)$. They essentially deal with the existence of models, that is with $I(\lambda, T)$ being $\geq 1$ or $0$.

Later, we will see that we can say lots of things on $I(\lambda, T)$ when $T$ is countable.

The last two chapters of Shelah’s book ‘Classification Theory’ (quite a few model theorists have called that book “The Bible”) are devoted to constructions converging to a theorem called the “Main Gap”. The theorem is quite out of reach of this course (a second course could lead to good chunks of it). However, you may try to understand what the theorem says at this point.

1.1 Main Gap and consequences

Theorem 1 (a version of the Main Gap) Let $T$ be a complete theory in a countable language. Then, if for some uncountable cardinal $\lambda$ we have $I(\lambda, T) < 2^\lambda$, we also have, for every ordinal $\alpha$

$$I(\mathbb{N}_\alpha, T) \leq \beth_{\omega_1}(\mathbb{N}_0 + |\alpha|).$$

Notice that this theorem divides countable complete first order theories in two “large classes”.

Exercise: explain in which sense this version of the MG divides theories into two classes.

Exercise: (easy) give as many examples as you can of theories in the “two classes”. (maybe hard) Redo this exercise every time you read about, invent, learn about a new FO theory.

2 Submodels and elementary submodels

We did not cover these in class:

1. $T$ categorical implies $T$ complete. Strange.
2. $T$ recursive and categorical implies $T$ decidable. Even stranger!
3. $T_{ord}, T_{ind}$ and $T_{ACF_p}$ are decidable.
4. $T_{ord} = Th(\mathbb{Q}, <), T_{ind} = Th(\text{Random Graph}), T_{ACF_0} = Th(\mathbb{C}, +, \times, 0, 1)$.

They now become an exercise for you.
3 Erdős-Rado

Usually, combinatorial proofs. The model theoretic proof is nicer: more insightful and conceptual. It also forces us to do interesting constructions that are fundamental in Mod Th.

Notation abuse in Mod Th: \( \bar{a} \in A, a \in A \) may mean \( \bar{a} \in A^n \), \( a \in A^n \), for appropriate \( n \).

Definition 2 Given \( M \) an \( L \)-structure, \( A \subset \|M\| \), \( b \in M \), the type of \( b \) over \( A \) in \( M \) is

\[
\text{tp}(b/A, M) = \{ \varphi(x, \bar{a}) | \varphi \in \text{Fml}(L), \bar{a} \in A, M \models \varphi[b, \bar{a}] \}.
\]

Examples: for \( L = \{+, \cdot, 0, 1\} \), look at \( \text{tp}(\sqrt{2}/\mathbb{Q}, \mathbb{C}) \), \( \text{tp}(\pi/\mathbb{Q}, \mathbb{C}) \), \( \text{tp}(3/\mathbb{Q}, \mathbb{C}) \). Notice \( \text{tp}(\pi/\mathbb{Q}, \mathbb{C}) = \text{tp}(e/\mathbb{Q}, \mathbb{C}) \)

Erdős notation:

\[
\lambda \rightarrow (\mu)^n_k
\]

means that for every coloring \( f \) of \( \lambda^n \) in \( \kappa \) many colors there is a monochromatic subset of \( \lambda \) of size \( \mu \).

Regular cardinals are usually defined as cardinals whose cofinality is themselves. Check that \( \kappa \) uncountable is regular iff \( \lambda \rightarrow (\lambda)^1_\kappa \) for every \( \kappa < \lambda \).

The Axiom of Choice implies that every successor (infinite) cardinal \( \mu^+ \) is regular.

Monotonicity of \( \lambda \rightarrow (\mu)^n_\kappa \):

\[
\lambda \rightarrow (\mu)^n_\kappa
\]

means that for every \( F : [\lambda]^{<\omega} \rightarrow \kappa \) there exists \( S \subset \lambda \) of cardinality \( \mu \) such that the image of \( [S]^{<\omega} \) under \( F \) is a singleton.

(Infinitary) Ramsey: \( \aleph_0 \rightarrow (\aleph_0)^2_2 \). Check finite Ramsey.

Sierpiński: \( \aleph_1 \not\rightarrow (\aleph_1)^2_2 \).

An uncountable cardinal \( \kappa \) is weakly compact when \( \kappa \rightarrow (\kappa)^2_\kappa \).

The status of the weakly compacts: they are (rather small) large cardinals!

Erdős-Rado: \( \forall n < \omega \forall \lambda \geq \aleph_0 \)

\[
\exists_n (\lambda)^+ \rightarrow (\lambda^+)^{n+1}_\lambda.
\]

Idea of the proof: induct on \( n < \omega \). Prove a ‘downward LS Lemma’ for realization of types (fill!). For \( n = 0 \), use that successor cardinals are regular. For the inductive step, do the following:

Pick infinite \( \lambda \), write \( \mu = \exists_n (\lambda) \). Have to show

\[
\exists_{n+1} (\lambda)^+ \rightarrow (\lambda^+)^{n+2}_\lambda.
\]
So pick $F : [(2\mu)^+]^{\mu+2} \rightarrow \lambda$ a coloring. We need to show there is a monochromatic $S \subset (2\mu)^+$ of size $\lambda^+$.

Look at the model

$$N = ((2\mu)^+, <, F, i)_{i < \lambda}.$$ 

Inside, build a continuous chain of $\mu^+$-many elementary submodels $N_i \prec N$ each of size at most $2\mu$ (so that we may apply the ind. hyp.) and closed under realizations of types of elements of $N$ over subsets $B$ of each predecessor, of size $\leq \mu$.

Take some $\alpha^* < (2\mu)^+$ outside the sup of the union of the chain (check it exists!)... and pick elements $a_i \in N_{i+1}$ such that

$$tp(a_i/A_i, N) = tp(\alpha^*/A_i, N),$$

where $A_i = \{a_j | j < i\}$.

Work inside $A$... Main point in previous paragraph is the relative indiscernibility of the $a_i$'s. Need of course a lemma that guarantees that this inductive step can be done.

**Lemma 3** Let $N$ be a model, $X \subset \|N\|$ and $\kappa$ such that $2^{|X|+|L|+\aleph_0} < \kappa < \|N\|$. Then there exists $M \prec N$ of cardinality $\kappa$ such that $M$ realizes all types of elements of $N$ with parameters in $X$.

This of course is almost enough, as in the construction we definitely wish to stay “only up to” $\mu$. We need to do this “uniformly” for subsets of $X$.