Set Theory – Spring 2003 – Homework 1

Name:
Due January 31

1. Which ZF axioms does the model $\mathfrak{A} = (A, E)$ satisfy? Explain four of them.

2. Let $\mathfrak{A} = \{X \subset \omega|X$ is finite of cofinite$, and let $\preceq$ be the following binary relation on $\mathfrak{A}$:

\[
X \preceq Y \iff \begin{cases} 
\text{either } X = Y \\
\text{or } #(X) \neq #(Y) \in \omega \\
\text{or } #(\omega \setminus Y) \neq #(\omega \setminus X) \in \omega
\end{cases}
\]

(a) Prove that $\preceq$ orders $\mathfrak{A}$ in a non total way.

(b) For $\mathfrak{B} = \{X \subset \omega|\#(X) \in \omega\}$ and $\mathfrak{C} = \{X \subset \omega|\#(\omega \setminus X)\text{ is a power of } 2\}$, compute $\max \mathfrak{B}$, $\min \mathfrak{C}$, $\text{Minim} \mathfrak{B}$, $\mathfrak{E}$, $\text{sup} \mathfrak{B}$, where $\text{Minim} \mathfrak{B}$ denotes the set of minimal elements of $\mathfrak{B}$ and $\mathfrak{E}$ denotes the set of upper bounds of $\mathfrak{C}$.

3. In $L^1 = \{f : [0, 1] \to \mathbb{R}| \int_0^1 f dx \in \mathbb{R}\}$, consider the equivalence relation

\[f \sim g \iff \int_0^1 (f - g) dx = 0.\]

(a) Find a \textbf{continuous} function in the equivalence class of $\chi_\mathbb{Q}$ (the characteristic function of $\mathbb{Q}$).

(b) Is it true that if $f$ is in the class of the constant function of value 1, then it must be positive?

(c) If we change the last 0 in the definition of $\sim$ to some $\epsilon > 0$, do we still have an equivalence relation?
4. What is wrong in the following proof by induction?

**Theorem:** All elements of any set are the same.

**Proof:** Induction on the size of the set. Clearly true for sets of size 1. Suppose the result holds for all sets of size \( n \). Take a set \( a \) with \( n + 1 \) elements, say \( a = \{y_1, y_2, \ldots, y_{n-1}, y_n, y_{n+1}\} \). Then the sets \( \{y_1, \ldots, y_{n-1}, y_n\} \) and \( \{y_1, \ldots, y_{n-1}, y_{n+1}\} \) both have \( n \) elements, so (by induction hypothesis) all the \( y_i \) are equal to \( y_1 \).

5. Prove that if \( |S| \geq 2 \), then \( |T| \leq |S|^T \).

6. Prove that \( \mathfrak{A} = (\mathbb{R}, +, <, 0) \) is isomorphic to \( \mathfrak{B} = (\mathbb{R}^+, \cdot, <, 1) \).

7. Prove that \([0, 4]\) is equipotent to \([1, 3]\).

8. Consider the plane \( \mathbb{R}^2 \), and the order relation

\[
(x, y) \preceq (z, w) \iff \begin{cases}
\text{either } z = 0, w > 0, (x \neq 0 \text{ or } (x = 0 \text{ and } y < 0)) \\
\text{or } x = 0, y < 0, (z \neq 0 \text{ or } (z = 0 \text{ and } w > 0)) \\
\text{or } (x, y) = (z, w) \\
\text{or } z \neq 0 \text{ and } x \neq 0 \text{ and } \frac{y}{x} < \frac{w}{z}.
\end{cases}
\]

Find maximal elements, minimal elements, max, min, upper and lower bounds, sups and infs of \( S^1, \mathbb{R} \times \{0\} \) and \( \{(x, y)|2x^2 + y^2 = 1\} \).