Basic Math

Tim Flaherty

April 7, 2004

Abstract

In this lab we examine some of the basic math type-setting found in a typical Calculus text.

1 Functions

We assume that students are familiar with the fundamentals regarding functions of a real variable. Concepts such as domain, range, evaluation, composition, and so on should be well understood. Also, students should be familiar with inverses of one-to-one functions. Various special functions, such as the trigonometric functions sin, cos, tan, cot, sec, csc, the exponential exp, and logarithm log also need to be known.

1.1 Limits

The limit is necessary to define continuity, differentiation, and integration. Indeed, Calculus would not be possible without the study of limits. We have the definition of a limit.

Definition 1.1 We say that $\lim_{x \to a} f(x) = L$ if given any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

1.2 Continuity

A continuous function is one whose graph has no “breaks” or “jumps”. Also, a continuous function is one in which small changes in the independent variable can induce only a small change in the dependent variable. The formal definition requires the limit. First we define continuity at a point.

Definition 1.2 A function $f$ is continuous at $a$ if
1. \( f \) is defined at \( a \), so \( f(a) \) exists,

2. \( \lim_{x \to a} f(x) \) exists,

3. the above quantities are the same, \( f(a) = \lim_{x \to a} f(x) \).

We call a function continuous if it is continuous at all numbers in its domain.

2 Derivatives

The derivative of a function measures the instantaneous rate of change of the function at some number. Again, the definition requires the use of a limit.

**Definition 2.1** A function \( f \) is **differentiable** at \( a \) if

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

exists. When this limit exists, we call the value of the limit the derivative of \( f \) at \( a \), and write this as \( f'(a) \).

**Ex. 2.2** Let \( f(x) = x^2 \). We’ll determine the derivative of \( f \) at \( a = 1 \).

\[
f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}
= \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1}
= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}
= \lim_{x \to 1} x + 1
= 2.
\]

Hence \( f'(1) = 2 \).

2.1 Differentiation Rules

First the power rule

\[
\frac{d}{dx} x^n = nx^{n-1}, \quad (1)
\]
and then the linearity of differentiation

\[(cf)'(x) = cf'(x),\]
\[(f + g)'(x) = f'(x) + g'(x)\]

are obtained. More complicated functions are differentiated with the product rule,

\[(fg)'(x) = f(x)g'(x) + f'(x)g(x),\]

the quotient rule,

\[\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2},\]

and the chain rule,

\[(f \circ g)'(x) = f(g(x))g'(x).\]

3 Integration

The definite integral is defined by taking a limit of Riemann sums

**Definition 3.1** Let \( f \) be a function defined on an interval \([a, b]\). For each natural number \( n \) define a partition \( P_n \) of \([a, b]\),

\[ a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b. \]

Let the norm of this partition be defined by

\[ ||P_n|| = \max_{1 \leq i \leq n} (x_i - x_{i-1}). \]

Let \( x_i^* \) be any number in the interval \([x_{i-1}, x_i]\). If

\[ \lim_{||P_n|| \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \]

exists for any choice of partition sequence \( P_n \) and sampling points \( x_i^* \), then we say \( f \) is integrable on \([a, b]\), with definite integral

\[ \int_{a}^{b} f(x) \, dx = \lim_{||P_n|| \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}), \]
and this limit is independent of the choice of \( P_n \) and \( x_i^* \).

This is a bit complicated, but fortunately we have a theorem that all continuous functions are integrable. The fundamental theorem of calculus simplifies integration in many problems.

**Theorem 3.2 The Fundamental Theorem of Calculus.** Suppose \( F'(x) = f(x) \) for all \( x \) in the interval \([a, b] \). Then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

**Ex. 3.3** We know that \( \frac{d}{dx} \sin(x) = \cos(x) \), hence

\[
\int_0^\frac{\pi}{2} \cos x \, dx = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1.
\]

In vector calculus we encounter double, triple, and path integrals. \LaTeX{} has symbols for these

\[
\iint, \iiint, \int
d\vec{f},
\]

also we compute partial derivatives, the gradient, curl, and divergence

\[
f_x = \frac{\partial f}{\partial x},
\nabla(f) = \text{grad}(f),
\nabla \times \mathbf{F} = \text{curl}(\mathbf{F}),
\nabla \cdot \mathbf{F} = \text{div}(\mathbf{F})
\]

**Theorem 3.4 Stokes Theorem.** Suppose \( S \) is a smooth orientable surface with boundary curve \( C \) oriented according to the right-hand rule convention. Let \( \mathbf{F} \) be a field defined and smooth on \( S \). Then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.
\]