This document contains a list of the important definitions and theorems that have been covered thus far in the course. It is not a complete listing of what has happened in lecture. The sections from the book that correspond with each topic are also given.

Following the list of important definitions and theorems you will find a collection of review exercises and sample test questions.

1. **Counting:** Sections 3.1, 3.2, and 3.3.

   **Definition 1.** If $X$ is a finite set then $2^X$ is the collection of all subsets of $X$; to be precise,
   
   $$2^X = \{A : A \subseteq X\}.$$  

   **Claim 2.** If $X$ is a finite set then $|2^X| = 2^{|X|}$. 

   **Definition 3.** Let $X = \{1, \ldots, n\}$. The characteristic vector of a set $A \subseteq X$ is $1_A = (y_1, \ldots, y_n)$ where
   
   $$y_i = \begin{cases} 
   1 & \text{if } i \in A \\
   0 & \text{if } i \notin A.
   \end{cases}$$

   **Definition 4.** Let $X$ be a finite set. For $k \in \mathbb{N}$ we define $\binom{X}{k}$ to be the collection of all $k$-element subsets of $X$; that is,
   
   $$\binom{X}{k} = \{A \subseteq X : |A| = k\}.$$ 

   **Claim 5.** If $X$ is a finite set and $k \in \mathbb{N}$ then
   
   $$\left| \binom{X}{k} \right| = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$ 

   **Claim 6.** If $n \in \mathbb{N}$ then
   
   $$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$ 

   **Claim 7.** If $n \in \mathbb{N}$ and $0 \leq k \leq n$ is an integer then
   
   $$\binom{n}{k} = \binom{n}{n-k}.$$ 

   **Claim 8.** If $n \in \mathbb{N}$ and $1 \leq k \leq n$ is an integer then
   
   $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$ 

   **Definition 9.** Let $X$ and $Y$ be sets. $X^Y$ is the collection of functions from $Y$ to $X$; that is,
   
   $$X^Y = \{f : Y \to X\}.$$
There is a correspondence between functions in \( X^Y \) and strings of elements of \( X \) indexed by \( Y \). If \(|Y| = \{1, \ldots, r\}\) then the string \((x_1, \ldots, x_r)\) corresponds to the function \( f \) such that \( f(i) = x_i \) for \( i = 1, \ldots, r \).

**Definition 10.** For \( n, k \in \mathbb{P} \), \( S(n, k) \) is defined to be the number of partitions of a set of size \( n \) into \( k \) non-empty parts. These are called the Stirling numbers of the second kind.

**Claim 11.** Let \( X \) and \( Y \) be finite sets such that \(|X| = n \) and \(|Y| = r \). We have
\[
|X^Y| = |X|^{|Y|} = n^r
\]
\[
|\{ f \in X^Y : f \text{ is an injection} \}| = n(n-1) \cdots (n-r+1)
\]
\[
|\{ f \in X^Y : f \text{ is a surjection} \}| = S(r, n)n!.
\]
Note that if \( n < r \) then the number of injections is 0. Furthermore, if \( r < n \) then the number of surjections is 0.

**Claim 12.** Let \( Z \) be a collection of \( k \) indistinguishable objects. Let \( X \) be a set of \( n \) distinguishable labels. We consider labellings of the objects of \( Z \) with the elements of \( X \). We consider two such labellings distinct if there is a label in \( X \) that appears a different number of times in the two labellings. So, there is correspondence between the set of distinct labeling and the set
\[
\{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{N} \text{ and } x_1 + \cdots + x_n = k\}.
\]
(In this correspondence \( x_i \) is the number of times label \( i \) is used.) We have
\[
\text{total number of labellings} = \binom{n+k-1}{n-1}
\]
\[
\text{number of labellings in which every label is different} = \binom{n}{k}
\]
\[
\text{number of labellings in which every label appears} = \binom{k-1}{n-1}
\]

2. **Counting and polynomials; generating functions:** Section 12.1.

**Notation 13.** If \( f(x) \) is a polynomial in \( x \) then \([x^n]f(x)\) is the coefficient of \( x^n \) in \( f \). For example, if \( f(x) = 1 + 2x^2 + 3x^3 + 5x^4 \) then \([x^2]f(x) = 22\).

**Note 14.** If \( f_1(x), f_2(x), \ldots, f_k(x) \) are polynomials in \( x \) then
\[
[x^n] \prod_{i=1}^{k} f_i(x) = \sum_{i_1, \ldots, i_k \in \mathbb{N}} [x^{i_1}]f_1(x) \cdot [x^{i_2}]f_2(x) \cdots [x^{i_k}]f_k(x).
\]

**Theorem 15 (Binomial Theorem).** For any \( n \in \mathbb{P} \) we have
\[
(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.
\]
(Note: this holds over any field.)
Definition 16. Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The generating function for this sequence is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$ 

Note 17. A generating function can be viewed as either

(i) a function of $x$ (when we have convergence), or

(ii) a formal object with addition and multiplication.

Note 18. 

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$ 

This can be viewed as either

(i) A fact for formal power series that follows from noting that 

$$(1-x)(1 + x + x^2 + \cdots) = 1,$$ or 

(ii) the power series for the function $1/(1-x)$, which converges for $|x| < 1$.

3. Estimates: Sections 3.4, 3.5 and 3.6.

Definition 19. Let $f : \mathbb{N} \rightarrow \mathbb{R}$ and $g : \mathbb{N} \rightarrow \mathbb{R}$ be functions (we usually assume $f(n), g(n) \geq 0$ for all $n$). We write $f \sim g$ if 

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$ 

Theorem 20 (Stirling’s Formula).

$$n! \sim \left( \frac{n}{e} \right)^n \sqrt{2\pi n}.$$ 

Note 21. (a) 

$$\binom{n}{n/2} \sim \sqrt{\frac{2}{\pi n}} 2^n$$

(b) If $k < \left\lfloor \frac{n}{2} \right\rfloor$ then 

$$\binom{n}{k+1} \binom{n}{k} = \frac{n-k}{k+1} \geq 1.$$ 

It follows that 

$$\binom{n}{0} \leq \binom{n}{1} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} \geq \cdots \geq \binom{n}{n-1} \geq \binom{n}{n}.$$ 

Claim 22. For any $n \in \mathbb{P}$ and $1 \leq k \leq n$ we have 

$$\binom{n}{k} \leq \left( \frac{ne}{k} \right)^k.$$
4. Inclusion/Exclusion: Sections 3.7 and 3.8.

**Theorem 23** (Inclusion/Exclusion Formula). Let $A_1, A_2, \ldots, A_n$ be subsets of a set $\Omega$. For $S \subseteq \{1, \ldots, n\}$ define

$$A_S = \begin{cases} \Omega & \text{if } S = \emptyset \\ \cap_{i \in S} A_i & \text{if } S \neq \emptyset. \end{cases}$$

We have

$$\left| \Omega \setminus \left( \bigcup_{i=1}^n A_i \right) \right| = \left| \bigcap_{i=1}^n A_i \right| = \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|S|} |A_S|.$$ 

**Definition 24.** Let $X$ be a finite set. A permutation $\sigma : X \to X$ is a derangement if $\sigma(x) \neq x$ for all $x \in X$.

**Theorem 25.** Let $D_n$ be the number of derangements of the set $\{1, 2, \ldots, n\}$. We have

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$ 

**Corollary 26.** $D_n \sim \frac{n!}{e}$.

**Theorem 27** (A more general form of inclusion/exclusion). Let $\Omega$ be a finite set, $f : \Omega \to \mathbb{R}$ be a function, and $A_1, \ldots, A_n$ be a collection of subsets of $\Omega$. For $S \subseteq \{1, \ldots, n\}$ set

$$A_S = \begin{cases} \cap_{i \in S} A_i & \text{if } S \neq \emptyset \\ \Omega & \text{if } S = \emptyset. \end{cases}$$

For $X \subseteq \Omega$ let

$$f(X) = \sum_{x \in X} f(x).$$

We have

$$f \left( \Omega \setminus \left( \bigcup_{i=1}^n A_i \right) \right) = \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|S|} f(A_S).$$

**Note 28.** Theorem 27 can be used to formulate a version of inclusion/exclusion for probability spaces.

**Review Exercises:** Working the following problems should help in preparation for the test. They are not necessarily ‘sample’ test questions.

1. Give a combinatorial proof of the following identity:

$$k^n = \sum_{i=1}^{n} \binom{k}{i} i! S(n, i).$$
2. Give a combinatorial proof of the following identity:
\[ \sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}. \]

3. Determine the number of strings of \( p \) 0’s and \( q \) 1’s where each pair of 1’s is separated by at least 2 0’s.

4. A word over the alphabet \( \{a, b, c, \ldots, z\} \) is called increasing if its letters, apart from repetitions, appear in alphabetical order. For example \( aabcc \) is increasing but \( abze \) is not. How many increasing words are there of length \( n \)?

5. Let \( h_0, h_1, h_2, \ldots \) be the sequence defined by \( h_n = \binom{n}{3} \). Determine the generating function for this sequence.

6. How many graphs on the vertex set \( \{1, 2, \ldots, 2n\} \) are isomorphic to the graph consisting of \( n \) disjoint edges (i.e. the graph with edge set \( \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2n-1, 2n\}\} \))? Explain your answer.

7. Prove that for \( n = 1, 2, \ldots \) we have
\[ 2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1. \]

8. Use a combinatorial argument to find integers \( a, b, c \) such that
\[ m^3 = a \binom{m}{3} + b \binom{m}{2} + c \binom{m}{1}. \]

From the text: 3.1.2, 3.1.4, 3.3.4, 3.3.9, 3.3.19, 3.3.25, 3.4.3, 3.5.2, 3.5.11, 3.7.2, 3.7.5, 3.8.4, 3.8.6, 3.8.13, 12.1.2, 12.1.3
SAMPLE QUESTIONS:

1. Find all positive integers \(a > b > c\) for which
\[
\binom{a}{b} \binom{b}{c} = 2 \binom{a}{c}.
\]
Give a combinatorial justification for your answer.

2. Give a combinatorial proof of the following statement: If \(k < \left\lfloor \frac{n}{2} \right\rfloor\) then
\[
\binom{n}{k+1} \binom{n}{k} = \frac{n-k}{k+1}.
\]

3. Let \(Y = \{y_1, y_2, \ldots, y_{2n}\}\) and \(X = \{x_1, x_2, \ldots, x_n\}\). Let \(\Omega\) be the set of all functions \(f : Y \to X\). The number of functions \(f \in \Omega\) such that \(f(y_i) \neq x_i\) for \(i = 1, 2, \ldots, n\) is
\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j (n)^{2n-j}.
\]

4. Let \(X = \{1, \ldots, n\}\). Determine the number of ordered pairs \((A, B)\) such that \(A \subseteq B \subseteq X\).

The test will include some questions in the following format: For each of the following statements, say whether the statement is true or false and give a short justification for your answer.

1. \(\binom{n}{10} \sim n^{10}/(10!)\).
2. \(n! \sim \sqrt{n(n/e)^n}\).
3. If \(n, k \in \mathbb{P}\) and \(0 \leq k \leq n\) then
\[
\binom{n}{k} \leq 2^n.
\]
4. If \(A\) and \(B\) are finite sets, \(f : A \to B\) is a surjection, and \(g : B \to A\) is a surjection then \(|A| = |B|\).
5. \(S(a, b) = \binom{a+b-1}{b-1}\).
(Recall that \(S(a, b)\) is the number of partitions of a set of size \(a\) into \(b\) nonempty parts.)
6. Let \(\epsilon > 0\) be a constant. There exists a constant \(\alpha > 0\) such that \(k < \alpha n\) implies
\[
\binom{n}{k} < (1 + \epsilon)^n
\]
for all \(n\).
7. If \(n \geq 3\) and \(1 < k < n\) then \(S(n, k) = S(n-1, k-1) + kS(n-1, k)\).
8. The number of ways to color a collection of \(n\) (indistinguishable) balls using 10 colors is \(\binom{n+10-1}{n-1}\).