1. Use Cayley’s formula to determine the number of trees on vertex set \{v_1, \ldots, v_n\} that contain the edge \{v_1, v_2\}.

2. Determine the number of the trees on vertex set \{v_1, \ldots, v_n\} in which every vertex has degree 1 or 3.

3. Let \(V = \{1, 2, \ldots, n\}\) and let \(G = (V, E)\) be a tree. Let \(A_1, A_2, \ldots, A_n\) be subsets of a finite set \(\Omega\). Prove that

\[
\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i| - \sum_{\{u,v\} \in E} |A_u \cap A_v|.
\]

4. Let \(F\) be a collection of subsets of \(\{1, \ldots, n\}\). We say that \(F\) is intersecting if every pair of sets in \(F\) intersect; to be precise, the collection \(F\) is intersecting if

\[
A, B \in F \implies A \cap B \neq \emptyset.
\]

(a) Prove that if \(F\) is an intersecting collection of subsets of \(\{1, \ldots, n\}\) then \(|F| \leq 2^{n-1}\).

(b) Give an example of an intersecting collection \(F\) of subsets of \(\{1, \ldots, n\}\) such that \(|F| = 2^{n-1}\).

5. Let \(F\) be a collection of subsets of \(\{1, 2, \ldots, n\}\). We say that \(F\) is semi-independent if it contains no three distinct sets \(A, B, C\) such that \(A \subset B \subset C\).

Adapt the proof of Sperner’s Theorem given in class to prove that if \(F\) is a semi-independent collection of subsets of an \(n\)-element set then

\[
|F| \leq 2 \left( \binom{n}{\lfloor n/2 \rfloor} \right).
\]
6. Consider the claim and proof given below. The Claim is true. Is the proof correct or incorrect? If the proof is incorrect then

(a) Explain the error in the proof and provide an example that illustrates the error.
(b) Give a correct proof.

**Claim.** If \( G = (V, E) \) is a connected graph and \( |V| = |E| \) then \( G \) contains exactly one cycle.

**Proof.** We go by induction on \( |V| \).

Base case. Note that \( \binom{|V|}{2} < |V| \) if \( |V| = 1 \) or \( |V| = 2 \). Therefore the smallest value of \( |V| \) we should consider is \( |V| = 3 \).

If \( G = (V, E) \) is a graph such that \( |V| = 3 \) and \( |E| = 3 \) then \( G \) is the complete graph on three vertices (i.e. a triangle). This graph has exactly one cycle.

Inductive step. Let \( n \geq 3 \) and let \( G = (V, E) \) be a connected graph with \( |V| = |E| = n \). Our inductive assumption is that \( G \) has exactly one cycle.

Let \( G' = (V', E') \) be a connected graph such that \( |E'| = |V'| = n + 1 \) which is an extension of \( G \) by one vertex. Formally, we have \( V \subset V' \) and \( E \subset E' \).

Let \( v \) be the one vertex in \( V' \) that is not in \( V \). Let \( e \in E' \) be an edge that touches the vertex \( v \) (formally \( v \in e \)). This edge is not in \( E \) (since \( v \) is not a vertex in \( G \)). Since \( E' \) has only one more edge than \( E \) we have \( E' = E \cup \{e\} \).

Consider a cycle in \( G' \). Either this cycle contains \( v \) or it does not. Since the degree of \( v \) in \( G' \) is 1 there are no cycles in \( G' \) that contain \( v \). If the cycle does not contain \( v \) then it is a cycle in \( G \). By our inductive assumption there is exactly one cycle of this form. Therefore \( G' \) contains exactly one cycle. 

\( \square \)