1. Find all positive integers $a > b > c$ for which
\[
\binom{a}{b} \binom{b}{c} = 2 \binom{a}{c}.
\]
Give a combinatorial justification for your answer (i.e. appeal to the fact that $\binom{n}{k}$ gives the number of subsets of an $n$-element set having exactly $k$ elements).

2. If $m$ indistinguishable 6-sided die are rolled how many possible outcomes are there?

3. Give a combinatorial proof of the following identity:
\[
n2^{n-1} = \sum_{i=1}^{n} \binom{n}{i}.
\]

4. A function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is **monotone** if $i < j$ implies $f(i) \leq f(j)$. How many monotone functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there?

5. A **3-uniform hypergraph** on vertex set $V$ is a collection of $H$ of 3-element subsets of $V$. For example
\[
H = \{\{2, 3, 5\}, \{1, 4, 7\}, \{1, 2, 4\}, \{5, 6, 7\}\}
\]
is a 3-uniform hypergraph on vertex set $V = \{1, 2, \ldots, 7\}$. How many different 3-uniform hypergraphs are there on vertex set $V = \{1, \ldots, 100\}$?

6. Let $S$ be the set of sequences $(x_1, \ldots, x_n)$ of non-negative integers such that $\sum_{i=1}^{n} x_i = m$. Let $T$ be the set of sequences $(y_1, \ldots, y_{m+1})$ of non-negative integers such that $\sum_{j=1}^{m+1} y_j = n - 1$. Give a **bijective** proof of the identity $|S| = |T|.$

7. (a) Give a combinatorial proof of the following equation
\[
\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.
\]
If you make use of a partition, explicitly state the equivalence relation that gives the partition.

(b) Use (a) to calculate $\sum_{i=1}^{n} i^2$. **Hint**: Consider $k = 2$.

(c) Use (a) to calculate $\sum_{i=1}^{n} i^3$. 