Abstract—The problems of stability and optimal control for stochastic difference equations are receiving important attention now (see for example [1-6]). In this paper, the necessary optimality condition for nonlinear stochastic difference second kind Volterra equation are constructed. For stochastic integral-functional equations analogous results were obtained in [7].

Keywords—Optimal control, Stochastic difference equations, Volterra difference equations.

1. THE STATEMENT OF THE PROBLEM

Consider the optimal control problem \( \{x_u, J(u), U\} \) where \( x_u \) is the motion trajectory, \( J(u) \) is the cost functional, and \( U \) is the set of the admissible controls.

The control \( u_0 \), for which the cost functional \( J(u) \) is minimal, i.e., \( J(u_0) = \min_{u \in U} J(u) \), is called optimal control.

Let \( u_\varepsilon, \varepsilon \geq 0 \) be the admissible control, for which there exists the limit

\[
J_0(u_0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [J(u_\varepsilon) - J(u_0)].
\] (1)

Obviously, the inequality \( J_0(u_0) \geq 0 \) is a necessary condition for the optimality of the control \( u_0 \). The existence of the limit (1) and the form of the necessary optimality condition are depended from the form of the control \( u_\varepsilon \). Let be, for example,

\[
u_\varepsilon = u_0 + \varepsilon v, \quad u_0, v \in U.
\] (2)

In this case \( J_0(u_0) = J_0(u_0, v) \) is Gâteaux differential.

In particular, if \( J_0(u_0, v) \) is linear with respect to \( v \) then \( J_0(u_0, v) = \langle J_0(u_0), v \rangle \) where \( J_0(u_0) \) is Gâteaux derivative. In the linear case the inequality \( \langle J_0(u_0), v \rangle \geq 0 \) is equivalent to the equation \( J_0(u_0) = 0 \) which has the unique solution \( u_0 \) \([5] \).

Using the calculation of the limit (1), (2) a necessary optimality condition for the nonlinear optimal control problem is obtained below.
2. MAIN RESULT

Consider the optimal control problem for the difference equation

\[ x(i+1) = \eta(i+1) + \Phi(i+1, x_{i+1}) + \sum_{j=0}^{i} a(i,j, x_j, u(j)) + \sum_{j=0}^{i} b(i,j, x_j, u(j)) \xi(j), \quad x(0) = \eta(0), \] (3)

and the cost functional

\[ J(u) = E \left[ F(x_N) + \sum_{j=0}^{N-1} G(j, x_j, u(j)) \right]. \] (4)

Let \( \{\Omega, \sigma, P\} \) be a probability space, \( i \in Z = \{0, 1, \ldots, N\} \) be a discrete time, \( f_i \in \sigma, i \in Z, \) be a family of \( \sigma \)-algebras, \( E \) be a mathematical expectation, \( H \) be a space of \( f_i \)-adapted functions \( x(i) \in \mathbb{R}^n, i \in Z, \) such that

\[ ||x||_H^2 = \max_{j \leq i} E[|x(j)|^2] < \infty. \]

Let \( x_i \) be the trajectory of the process \( x(j), j \leq i, \eta(i) \) be the \( f_i \)-adapted random values, \( ||\eta||_N < \infty, \xi(i) \) be \( f_{i+1} \)-adapted random values, which independent from each others and from \( \eta(i), \)

\[ E\xi(i) = 0, E[|\xi(i)|^2] = 1, \Phi(i, \varphi) \in \mathbb{R}^n, a(i, j, \varphi, u) \in \mathbb{R}^n, b(i, j, \varphi, u) \in \mathbb{R}^n, 0 \leq j \leq i \leq N, \]

\[ u(i) \in \mathbb{R}^m, \varphi \in H. \]

Arbitrary \( f_i \)-adapted function \( u(i) \in \mathbb{R}^m \) with a finite norm \( ||u||_N < \infty, \) is called admissible control, \( U \) is the set of the admissible controls.

It is supposed that \( F(\varphi) \) depends from all values of the function \( \varphi(j) \) by \( j = 0, 1, \ldots, N, \Phi(i, \varphi) \)

depends from values of the function \( \varphi(j) \) by \( j = 0, 1, \ldots, i \) only, the functionals \( a(i, j, \varphi, u), \)

\( b(i, j, \varphi, u), G(j, \varphi, u) \) depend from values of the function \( \varphi(l) \) by \( l = 0, 1, \ldots, j \) only and satisfy

to the conditions

\[ |\Phi(i, \varphi)| \leq \sum_{j=0}^{i} (1 + |\varphi(j)|) K_0(j), \] (5)

\[ |a(i, j, \varphi, u)|^2 + |b(i, j, \varphi, u)|^2 \leq \sum_{l=0}^{j} (1 + |u|^2 + |\varphi(l)|^2) K_1(l), \] (6)

\[ |\Phi(i, \varphi_1) - \Phi(i, \varphi_2)| \leq \sum_{j=0}^{i} |\varphi_1(j) - \varphi_2(j)| K_0(j), \] (7)

\[ |\nabla \Phi(i, \varphi_1, \varphi)| \leq \sum_{j=0}^{i} |\varphi(j)| K_0(j), \] (8)

\[ |\nabla a(i, j, \varphi_1, u)\varphi|^2 + |\nabla b(i, j, \varphi_1, u)\varphi|^2 \leq \sum_{l=0}^{j} |\varphi(l)|^2 K_1(l), \] (9)

\[ |\nabla u a(i, j, \varphi_1, u)\varphi|^2 + |\nabla u b(i, j, \varphi, u)\varphi|^2 \leq C, \] (10)

\[ |(\nabla \Phi(i, \varphi_1) - \nabla \Phi(i, \varphi_2))\varphi|^2 \leq \sum_{j=0}^{i} |\varphi_1(j) - \varphi_2(j)|^2 K_0(j), \] (11)
\[\begin{align*}
|\nabla a(i, j, \varphi_1, u_1) - \nabla a(i, j, \varphi_2, u_2)|^2 + |(\nabla b(i, j, \varphi_1, u_1) - \nabla b(i, j, \varphi_2, u_2))\varphi|^2 \\
\leq \sum_{l=0}^{j} \left[ |\varphi_1(l) - \varphi_2(l)|^2 + |u_1 - u_2|^2 \right] |\varphi(l)|^2 K_1(l),
\end{align*}\]

\[|\nabla_u a(i, j, \varphi, u_1) - \nabla_u a(i, j, \varphi, u_2)|^2 + |\nabla_u b(i, j, \varphi, u_1) - \nabla_u b(i, j, \varphi, u_2)|^2 \leq C|u_1 - u_2|.\]

Here \(\nabla\) is Gâteaux derivative with respect to \(\varphi\) and \(\nabla_u\) is derivative with respect to \(u\).

For the cost functional it is assumed that

\[|F(\varphi)| \leq \sum_{j=0}^{N} (1 + |\varphi(j)|^2) K_1(j),\]  

\[|G(i, \varphi, u)| \leq \sum_{j=0}^{i} (1 + |u|^2 + |\varphi(j)|^2) K_1(j),\]  

\[|\nabla F(\varphi_1, \varphi)| \leq \sum_{j=0}^{N} (1 + |\varphi_1(j)|) |\varphi(j)|^2 K_1(j),\]  

\[|\nabla G(i, \varphi_1, u, \varphi)| \leq \sum_{j=0}^{i} (1 + |u| + |\varphi_1(j)|) |\varphi(j)|^2 K_1(j),\]  

\[|\nabla F(\varphi_1) - \nabla F(\varphi_2, \varphi)| \leq \sum_{j=0}^{N} |\varphi_1(j) - \varphi_2(j)| |\varphi(j)| K_1(j),\]  

\[|\nabla G(i, \varphi_1, u_1) - \nabla G(i, \varphi_2, u_2, \varphi)| \leq \sum_{j=0}^{i} (|\varphi_1(j) - \varphi_2(j)| + |u_1 - u_2|) |\varphi(j)| K_1(j),\]  

\[|\nabla_u G(i, \varphi, u_1) - \nabla_u G(i, \varphi, u_2)| \leq C|u_1 - u_2|.\]

Here \(\max_{i \in \mathbb{Z}} |K_0(i)| < 1.\)

**Definition.** The matrix \(b(i, j)\) is called the resolvent of the kernel \(a(i, j)\) if the solution of the equation

\[x(i + 1) = \eta(i + 1) + \sum_{j=0}^{i} a(i, j)x(j), \quad x(0) = \eta(0),\]

can be represented in the form

\[x(i + 1) = \eta(i + 1) + \sum_{j=0}^{i} b(i, j)\eta(j).\]

The kernel and the resolvent are connected by the relations:

\[b(i, j) = a(i, j) + \sum_{k=j+1}^{i} b(i, k)a(k - 1, j),\]

\[b(i, j) = a(i, j) + \sum_{k=j+1}^{i} a(i, k)b(k - 1, j).\]
THEOREM 1. Let the conditions (5)-(20) hold. Then the limit (1),(2) for the problem (3),(4) there exists and equals

\[ J_0(u_0) = E \left[ (F_0(N), q_0N) + \sum_{j=0}^{N-1} ((G_0(j), q_0j) + v'(j)q_0(j)) \right]. \]  

(21)

Here \( q_0(j) \) is the solution of the stochastic equation

\[ q_0(i + 1) = \eta_0(i + 1) + \Phi_0(i + 1)q_0,i+1 + \sum_{j=0}^{i} A_0(i, j)q_0j + \sum_{j=0}^{i} B_0(i, j)q_0j \xi(j), \quad q_0(0) = 0, \]  

(22)

where

\[ \eta_0(i) = \sum_{j=0}^{i} a_0(i, j)v(j) + \sum_{j=0}^{i} b_0(i, j)v(j)\xi(j), \]

\[ F_0(N) = \nabla F(x_{0N}), \quad \Phi_0(i) = \nabla \Phi(i, x_{0i}), \]

\[ G_0(j) = \nabla G(j, x_{0j}, u_0(j)), \quad g_0(j) = \nabla_u G(j, x_{0j}, u_0(j)), \]

\[ A_0(i, j) = \nabla a(i, j, x_{0j}, u_0(j)), \quad a_0(i, j) = \nabla_u a(i, j, x_{0j}, u_0(j)), \]

\[ B_0(i, j) = \nabla b(i, j, x_{0j}, u_0(j)), \quad b_0(i, j) = \nabla_u b(i, j, x_{0j}, u_0(j)). \]

For the proving of the theorem we need the auxiliary statements.

3. AUXILIARY STATEMENTS

LEMMA 1. Let

\[ 0 \leq z(0) \leq z(1) \leq \cdots \leq z(N - 1), \]

\[ y(i) \geq 0, \quad i = 1, 2, \ldots, N, \quad y(1) \leq C z(0), \]

\[ y(i + 1) \leq C \left[ z(i) + \sum_{j=1}^{i} y(j) \right]. \]

Then there exists the constant \( C_1 > 0 \) independent of \( z(i), y(i) \) and such that

\[ y(i + 1) \leq C_1 z(i), \quad i = 0, 1, \ldots, N - 1. \]

PROOF. Using the mathematical induction method let us supposed that \( y(j) \leq C z(j - 1) \) for \( j = 1, \ldots, i \). In particular, it holds by \( i = 1 \). Let us prove it for \( i + 1 \). Since \( z(j - 1) \leq z(i) \) by \( j \leq i \) then

\[ y(i + 1) \leq C \left[ z(i) + \sum_{j=1}^{i} C z(j - 1) \right] \leq C \left[ z(i) + \sum_{j=1}^{i} C z(i) \right] \leq C_1 z(i), \]

where \( C_1 = C(1 + CN) \). The lemma is proved.

LEMMA 2. Let \( x_\varepsilon(j) \) be the solution of the equation (3) by the control \( u_\varepsilon(j) \),

\[ q_\varepsilon(i) = \frac{1}{\varepsilon} [x_\varepsilon(i) - x_0(i)], \]  

(24)

and the conditions (7)-(12) hold. Then \( q_\varepsilon \in H \) uniformly for \( \varepsilon \geq 0 \).
PROOF. From (3), it follows

\[
q_\varepsilon(i + 1) = \frac{1}{\varepsilon} \left[ \Phi(i + 1, x_{\varepsilon,i+1}) - \Phi(i + 1, x_{0,i+1}) \right] + \frac{1}{\varepsilon} \sum_{j=0}^{i} \left[ a(i, j, x_{\varepsilon,j}, u_{\varepsilon}(j)) - a(i, j, x_{0,j}, u_{0}(j)) \right] \\
+ \frac{1}{\varepsilon} \sum_{j=0}^{i} \left[ b(i, j, x_{\varepsilon,j}, u_{\varepsilon}(j)) - b(i, j, x_{0,j}, u_{0}(j)) \right] \xi(j).
\]

Let

\[
\lambda^\varepsilon(i) = x_{0}(i) + \tau e q_{\varepsilon}(i), \quad u^\varepsilon(i) = u_{0}(i) + \tau e v(i), \quad \Phi_{\varepsilon}(i) = \frac{1}{0} \nabla \Phi(i, \lambda^\varepsilon) \, d\tau,
\]

\[
A_{\varepsilon}(i, j) = \int_{0}^{1} \nabla a(i, j, \lambda^\varepsilon(j), u_{\varepsilon}(j)) \, d\tau, \quad a_{\varepsilon}(i, j) = \int_{0}^{1} \nabla u a(i, j, x_{0,j}, u_{\varepsilon}(j)) \, d\tau,
\]

\[
B_{\varepsilon}(i, j) = \int_{0}^{1} \nabla b(i, j, \lambda^\varepsilon(j), u_{\varepsilon}(j)) \, d\tau, \quad b_{\varepsilon}(i, j) = \int_{0}^{1} \nabla u b(i, j, x_{0,j}, u_{\varepsilon}(j)) \, d\tau.
\]

Then

\[
q_{\varepsilon}(i + 1) = \Phi_{\varepsilon}(i + 1) q_{\varepsilon,i+1} + \sum_{j=0}^{i} A_{\varepsilon}(i, j) q_{\varepsilon,j} + \sum_{j=0}^{i} B_{\varepsilon}(i, j) q_{\varepsilon,j} \xi(j) \\
+ \sum_{j=0}^{i} a_{\varepsilon}(i, j) v(j) + \sum_{j=0}^{i} b_{\varepsilon}(i, j) v(j) \xi(j).
\]

(25)

From (9),(25) it follows

\[
|q_{\varepsilon}(i + 1)|(1 - K_{0}(i + 1)) \leq \sum_{j=0}^{i} |q_{\varepsilon}(j)| K_{0}(j) + \sum_{j=0}^{i} |A_{\varepsilon}(i, j) q_{\varepsilon,j}| \\
+ \sum_{j=0}^{i} B_{\varepsilon}(i, j) q_{\varepsilon,j} \xi(j) \right| + \sum_{j=0}^{i} |a_{\varepsilon}(i, j) v(j)| + \sum_{j=0}^{i} b_{\varepsilon}(i, j) v(j) \xi(j) \right|.
\]

Squaring this expression and calculating the mathematical expectation by virtue of (9),(10) we obtain

\[
\mathbb{E}|q_{\varepsilon}(i + 1)|^2 \leq C \left[ \sum_{j=0}^{i} \mathbb{E}|q_{\varepsilon}(j)|^2 K_{0}(j) + \sum_{j=0}^{i} \sum_{l=0}^{j} \mathbb{E}|q_{\varepsilon}(l)|^2 K_{1}(l) + \sum_{j=0}^{i} \mathbb{E}|v(j)|^2 \right] \\
= C \left[ \sum_{j=0}^{i} \mathbb{E}|q_{\varepsilon}(j)|^2 (K_{0}(j) + (i - j + 1) K_{1}(j)) + \sum_{j=0}^{i} \mathbb{E}|v(j)|^2 \right].
\]

Let be \( y_{\varepsilon}(i) = \mathbb{E}|q_{\varepsilon}(i)|^2, \quad z(i) = \sum_{j=0}^{i} \mathbb{E}|v(j)|^2. \) Then for some \( C > 0 \) we obtain

\[
y_{\varepsilon}(i + 1) \leq C \left[ z(i) + \sum_{j=1}^{i} y(j) \right].
\]

(26)

Using Lemma 1 and \( \mathbb{E}|v(j)|^2 < \infty \) we obtain that \( q_{\varepsilon} \in H \) uniformly for \( \varepsilon \geq 0 \). The lemma is proved.
LEMMA 3. Let the conditions (5)-(13) hold. Then \( \lim_{\varepsilon \to 0} \|q_\varepsilon - q_0\|_1 = 0. \)

**Proof.** From (25) it follows

\[
q_\varepsilon(i + 1) = \Phi_\varepsilon(i + 1)q_\varepsilon,i+1 - \Phi_0(i + 1)q_0,i+1 + \sum_{j=0}^{i}[A_\varepsilon(i,j)q_\varepsilon,j - A_0(i,j)q_0,j]
\]

\[
+ \sum_{j=0}^{i}[B_\varepsilon(i,j)q_\varepsilon,j - B_0(i,j)q_0,j]\xi(j) + \sum_{j=0}^{i}(a_\varepsilon(i,j) - a_0(i,j))v(j) + \sum_{j=0}^{i}(b_\varepsilon(i,j) - b_0(i,j))v(j)\xi(j).
\]

Therefore

\[
q_\varepsilon(i + 1) = \Phi_0(i + 1)[q_\varepsilon - q_0]_i+1 + (\Phi_\varepsilon(i + 1) - \Phi_0(i + 1))q_\varepsilon,i+1 + \sum_{j=0}^{i}(A_\varepsilon(i,j)[q_\varepsilon - q_0]_j
\]

\[
+ (A_\varepsilon(i,j) - A_0(i,j))q_\varepsilon,j) + \sum_{j=0}^{i}(B_\varepsilon(i,j)[q_\varepsilon - q_0]_j + (B_\varepsilon(i,j) - B_0(i,j))q_\varepsilon,j)\xi(j)
\]

\[
+ \sum_{j=0}^{i}(a_\varepsilon(i,j) - a_0(i,j))v(j) + \sum_{j=0}^{i}(b_\varepsilon(i,j) - b_0(i,j))v(j)\xi(j).
\]

Using (8),(9) analogously (26) we have

\[
y_\varepsilon(i + 1) \leq C \left[ \sum_{j=0}^{5} \alpha_1(j) + \sum_{j=0}^{5} y_\varepsilon(j) \right].
\]

Here

\[
y_\varepsilon(i) = E[q_\varepsilon(i) - q_0(i)]^2, \quad \alpha_1(\varepsilon, i) = E[|\Phi_\varepsilon(i) - \Phi_0(i)|q_\varepsilon,j]^2,
\]

\[
\alpha_2(\varepsilon, i) = E[|A_\varepsilon(i,j) - A_0(i,j)|q_\varepsilon,j]^2, \quad \alpha_3(\varepsilon, i) = E[|B_\varepsilon(i,j) - B_0(i,j)|q_\varepsilon,j]^2,
\]

\[
\alpha_4(\varepsilon, i) = E[(a_\varepsilon(i,j) - a_0(i,j))v(j)]^2, \quad \alpha_5(\varepsilon, i) = E[(b_\varepsilon(i,j) - b_0(i,j))v(j)]^2.
\]

Let us prove that \( \lim_{\varepsilon \to 0} \alpha_1(\varepsilon, i) = 0 \) uniformly for \( i \in \mathbb{Z} \). Let be \( \chi^K(i) \) the indicator of the form

\[
q_\varepsilon(i) = q_\varepsilon(i)\chi^K(i) + q_\varepsilon(i)(1 - \chi^K(i)).
\]

We obtain

\[
\alpha_1(\varepsilon, i) = E \left[ (\Phi_\varepsilon(i) - \Phi_0(i))q_\varepsilon(i)\chi^K(i) + q_\varepsilon(i)(1 - \chi^K(i)) \right]^2
\]

\[
= E \left[ \Phi_\varepsilon(i)q_\varepsilon(i)\chi^K(i) - \Phi_0(i)q_\varepsilon(i)\chi^K(i) + (\Phi_\varepsilon(i) - \Phi_0(i))q_\varepsilon(i)(1 - \chi^K(i)) \right]^2
\]

\[
\leq 3 \left[ E \Phi_\varepsilon(i)[q_\varepsilon\chi^K(i)]^2 + E \Phi_0(i)[q_\varepsilon\chi^K(i)]^2 + E \Phi_\varepsilon(i) - \Phi_0(i) \left[ q_\varepsilon(1 - \chi^K(i)) \right]^2 \].
\]

Using (8),(11) we obtain

\[
\alpha_1(\varepsilon, i) \leq C \left[ \sum_{j=0}^{i} E[q_\varepsilon\chi^K(j)]^2 K_0(j)
\right.
\]

\[
+ \sum_{j=0}^{i} E[(q_\varepsilon(j) - x_0(j))^2] \left( (q_\varepsilon(1 - \chi^K(j)))_j \right | K_1(j) \right) \leq C \left[ \|q_\varepsilon\chi^K\|_N^2 + \varepsilon^2 K^2 \right].
\]
As \( q_\varepsilon \in H \) uniformly for \( \varepsilon \), then \( \lim_{K \to \infty} \| q_\varepsilon X^K \|_N = 0 \). Hence, for all \( \delta > 0 \), there exist \( K \), that \( \| q_\varepsilon X^K \|_N^2 < \delta (2C)^{-1} \). Fixing this \( K \), choose \( \varepsilon \) so that \( \varepsilon^2 K^2 < \delta (2C)^{-1} \). Therefore for all \( \delta > 0 \) there exists \( \varepsilon > 0 \) that \( \alpha_i(\varepsilon, i) < \delta \).

Analogously, it is proved that \( \lim_{\varepsilon \to 0} \alpha_i(\varepsilon, i) = 0 \), \( i = 2, \ldots, 5 \). From (27) it follows, that for all \( \delta > 0 \) there exists \( \varepsilon > 0 \), that

\[
y_\varepsilon(i + 1) \leq C \left[ \delta + \sum_{j=1}^4 y_\varepsilon(j) \right].
\]

From the Lemma 1, we obtain

\[
\| q_\varepsilon - q_0 \|_i \leq C_1 \delta.
\]

The lemma is proved.

4. LIMIT CALCULATION

From (1) it follows

\[
J_\varepsilon(u_0) = \frac{1}{\varepsilon} [J(u_\varepsilon) - J(u_0)] = E \left[ \frac{1}{\varepsilon} (F(x_\varepsilon N) - F(x_0 N)) \right] + \frac{1}{\varepsilon} \sum_{j=0}^{N-1} (G(j, x_\varepsilon j, u_\varepsilon(j)) - G(j, x_0 j, u_0(j)))
\]

\[
= E(F_\varepsilon(N), q_\varepsilon N) + \sum_{j=0}^{N-1} ((G_\varepsilon(j), q_\varepsilon j) + v'(j) g_\varepsilon(j))].
\]

Here

\[
F_\varepsilon(N) = \int_0^1 \nabla F(N, \lambda^{*}_\varepsilon) \, dt, \quad G_\varepsilon(j) = \int_0^1 \nabla G(j, \lambda^{*}_\varepsilon, u_\varepsilon(j)) \, dt,
\]

\[
g_\varepsilon(j) = \int_0^1 \nabla u G(j, x_0 j, u_\varepsilon(j)) \, dt.
\]

From here it follows

\[
J_\varepsilon(u_0) = E \left[ (F_0(N), q_\varepsilon N) + \sum_{j=0}^{N-1} (G_0(j), q_\varepsilon j) v'(j) g_0(j) + \sum_{j=0}^3 \beta_j(\varepsilon) \right],
\]

where

\[
\beta_1(\varepsilon) = E(F_\varepsilon(N) - F_0(N), q_\varepsilon N), \quad \beta_2(\varepsilon) = E \sum_{j=0}^{N-1} (G_\varepsilon(j) - G_0(j), q_\varepsilon j),
\]

\[
\beta_3(\varepsilon) = E \sum_{j=0}^{N-1} \beta'(j) |g_\varepsilon(j) - g_0(j)|.
\]

Let us prove that \( \lim_{\varepsilon \to 0} \beta_1(\varepsilon) = 0 \). From (18) it follows

\[
\beta_1(\varepsilon) \leq \sum_{j=0}^N E|x_\varepsilon(j) - x_0(j)| |q_\varepsilon(j)| K_1(j) = \varepsilon \sum_{j=0}^N E|q_\varepsilon(j)|^2 K_1(j) \leq C \varepsilon \| q_\varepsilon \|_N^2.
\]

Analogously, estimations for \( \beta_2(\varepsilon), \beta_3(\varepsilon) \) is true. From here we have

\[
\lim_{\varepsilon \to 0} |J_\varepsilon(u_0) - J_0(u_0)| = 0.
\]

The theorem is proved.
5. LINEAR-SQUARE OPTIMAL CONTROL PROBLEM

Consider the linear-square optimal control problem with the motion trajectory

\[ x(i + 1) = \eta(i + 1) + \sum_{j=0}^{i} a(i,j)x(j) + \sum_{j=0}^{i} b(i,j)u(j), \quad x(0) = \eta(0), \quad (28) \]

and the cost functional

\[ J(u) = E \left[ x'(N)F x(N) + \sum_{j=0}^{N-1} u'(j)G(j)u(j) \right]. \quad (29) \]

Here \( x(i) \in \mathbb{R}^n, u(i) \in \mathbb{R}^m, a(i,j) \) and \( b(i,j) \) are deterministic \( n \times n \) and \( n \times m \)-matrices, \( F \) is positive semidefinite matrix and \( G(j), j \in \mathbb{Z}, \) are positive definite matrices.

For the problem (28),(29) the limit (21) has a form

\[ J_0(u_0) = 2E \left[ x'_0(N)F q_0(N) + \sum_{j=0}^{i} u'_0(j)G(j)v(j) \right]. \]

Here \( x_0 \) is the solution of equation (28) by the control \( u_0 \) and \( q_0(i) \) is the solution of the equation

\[ q_0(i + 1) = \sum_{j=0}^{i} a(i,j)q_0(j) + \sum_{j=0}^{i} b(i,j)v(j), \quad q_0(0) = 0. \]

In the linear case the equation \( J_0(u_0) = 0 \) has the unique solution \( u_0 \) [5]

\[ u_0(j + 1) = \alpha(j + 1) + p(j + 1)\psi(N - 1, j, I)x_0(j + 1) + \sum_{k=0}^{j} \gamma(j,k)x_0(k), \quad j = 0, 1, \ldots, N - 2, \]

\[ u_0(0) = \alpha(0) + p(0) \left( 1 + \sum_{k=0}^{N-1} R(N - 1, k) \right) x_0(0). \]

Here \( I \) is identical matrix,

\[ p(j) = -G^{-1}(j)\psi'(N - 1, j, b(\cdot, j)) \times F \left[ I + \sum_{k=j}^{N-1} \psi(N - 1, k, b(\cdot, k))G^{-1}(k)\psi'(N - 1, k, b(\cdot, k))F \right]^{-1}, \]

\[ \alpha(j + 1) = p(j + 1)\psi(N - 1, j, \beta(\cdot, j + 1)) + \sum_{k=0}^{j} Q(j,k)\psi(N - 1, k - 1, \beta(\cdot, k)), \]

\[ \gamma(j,k) = p(j + 1)\psi(N - 1, j, a_j(\cdot, k)) + Q(j,k)p(k)\psi(N - 1, k - 1, I) \]
\[ + \sum_{l=k+1}^{j} Q(j,l)p(l)\psi(N - 1, l - 1, a_{l-1}(\cdot, k)). \]

\( Q(j,k) \) is the resolvent of the kernel \( p(j + 1)\psi(N - 1, j, b_j(\cdot, k)), \)

\[ a_j(i,k) = a(i,k) - a(j,k), \quad b_j(i,k) = b(i,k) - b(j,k), \quad \beta(i,j) = E_j\eta(i + 1) - \eta(j). \]

REMARK 1. If process \( \eta(i) \) is martingale then \( \beta(i,j) = 0, i \geq j, \) and therefore \( \alpha(i) = 0, i \in \mathbb{Z}. \)
REMARK 2. Analogously we can obtain the optimal control for system with noise by the control. For example, consider scalar equation

$$x(i + 1) = \eta(i + 1) + \sum_{j=0}^{i}[\beta(i,j) + \gamma(i,j)u(j)]\xi(j), \quad x(0) = \eta(0),$$

with the cost functional

$$J(u) = \mathbb{E}\left[ x^2(N) + \lambda \sum_{i=0}^{N-1} u^2(i) \right], \quad \lambda > 0.$$

This example was considered in [5].

REFERENCES