About one application of the general method of Lyapunov functionals construction

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SUMMARY


KEY WORDS: stochastic difference equations; delay; stability; method of Lyapunov functionals construction

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1. STATEMENT OF THE PROBLEM

Let $i$ be a discrete time, $i \in \mathbb{Z}_0 \cup \mathbb{Z}$, $\mathbb{Z}_0 = \{-h, \ldots, 0\}$, $\mathbb{Z} = \{0, 1, \ldots\}$, $h$ be a given non-negative number, process $x_i \in \mathbb{R}^n$ be a solution of the equation

$$x_{i+1} = F(i,x_{i-h}, \ldots, x_i) + \sum_{j=0}^{i} G(i,j,x_{i-h}, \ldots, x_j) \xi_j, \quad i \in \mathbb{Z},$$

$$x_i = \phi_i, \quad i \in \mathbb{Z}_0. \tag{1}$$

Here $F : \mathbb{Z} \ast S \Rightarrow \mathbb{R}^n$, $G : \mathbb{Z} \ast \mathbb{Z} \ast S \Rightarrow \mathbb{R}^n$, $S$ is a space of sequences with elements from $\mathbb{R}^n$. It is assumed that $F(i, \ldots)$ does not depend on $x_j$ for $j > i$, $G(i, j, \ldots)$ does not depend on $x_k$ for $k > j$ and $F(i, 0, \ldots, 0) = 0$, $G(i, j, 0, \ldots, 0) = 0$.

Let $\{\Omega, \sigma, \mathbb{P}\}$ be a basic probability space, $f_i \in \sigma$, $i \in \mathbb{Z}$, be a family of $\sigma$-algebras, $\xi_0, \xi_1, \ldots$ be a sequence of mutually independent random values, $\xi_i$ be $f_{i+1}$-adapted and independent on $f_i$, $\mathbb{E}\xi_i = 0$, $\mathbb{E}\xi_i^2 = 1$, $i \in \mathbb{Z}$.

**Definition 1.1**

The trivial solution of equation (1) is called $p$-stable, $p > 0$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbb{E}|x|^p < \varepsilon$, $i \in \mathbb{Z}$, if $\|\phi\|^p = \sup_{i \in \mathbb{Z}} \mathbb{E}|\phi(i)|^p < \delta$. If, besides, $\lim_{i \to \infty} \mathbb{E}|x_i|^p = 0$ then the trivial solution of equation (1) is called asymptotically $p$-stable. In particular, if $p = 2$ then the trivial solution of equation (1) is called asymptotically mean square stable.

**Theorem 1.1** (Kolmanovskii and Shaikhet [1])

Let there exists a non-negative functional $V_i = V(i,x_{i-h}, \ldots, x_i)$, for which the conditions

$$\mathbb{E}V(0,x_{-h}, \ldots, x_0) \leqslant c_1 \|\phi\|^p \tag{2}$$

$$\mathbb{E}\Delta V_i \leqslant -c_2 \mathbb{E}|x_i|^p, \quad i \in \mathbb{Z} \tag{3}$$

hold. Here $c_1 > 0$, $c_2 > 0$, $p > 0$ and

$$\Delta V_i = V(i+1,x_{i-h}, \ldots, x_{i+1}) - V(i,x_{i-h}, \ldots, x_i) \tag{4}$$

Then the trivial solution of equation (1) is asymptotically $p$-stable.

From this theorem it follows that a problem of stability investigation of some hereditary system can be reduced to the problem of construction of appropriate Lyapunov functionals. Below some formal procedure of Lyapunov functionals construction for equation type of (1) is described.

2. FORMAL PROCEDURE OF LYAPUNOV FUNCTIONALS CONSTRUCTION

The proposed procedure of Lyapunov functionals construction consists of four steps.

1. Represent the functionals $F$ and $G$ at the right-hand side of Equation (1) in the form

$$F(i,x_{i-h}, \ldots, x_i) = F_1(i,x_{i-t}, \ldots, x_i) + F_2(i,x_{i-h}, \ldots, x_i) + \Delta F_3(i,x_{i-h}, \ldots, x_i)$$

$$F_1(i,0, \ldots, 0) \equiv F_2(i,0, \ldots, 0) \equiv F_3(i,0, \ldots, 0) \equiv 0$$
Let us apply the proposed procedure for the investigation of the scalar equation under consideration.

2. Consider the auxiliary difference equation

\[ y_{i+1} = F_i(i, y_{i-\tau}, \ldots, y_i) + \sum_{j=0}^{i} G_i(i, j, y_{j-\tau}, \ldots, y_j)\xi_j, \quad i \in \mathbb{Z} \]  

(6)

and suppose that for this equation there exists a Lyapunov function \( v(i, y_{i-\tau}, \ldots, y_i) \), which satisfies the stability conditions of Theorem 1.1.

3. Consider Lyapunov functional \( V_i \) for initial Equation (1) in the form \( V_i = V_{i1} + V_{i2} \), where the main component \( V_{i1} \) has the form

\[ \sum_{j=0}^{i} \alpha_{i-j} x_j, \quad i \in \mathbb{Z}, \]

where \( \tau \geq 0 \) is a given integer, \( \Delta F_i(i, x_{-h}, \ldots, x_i) = F_i(i+1, x_{-h}, \ldots, x_i) - F_i(i, x_{-h}, \ldots, x_i) \).

4. Using obtained estimation for \( \Delta V_{i1} \), the additional component \( V_2 \) of the functional \( V_i = V_{i1} + V_{i2} \) can be chosen usually by some standard way.

This procedure is demonstrated in Reference [1] by considering a lot of concrete equations. But for equations type Volterra the parameter \( \tau \) in representation (5) takes the values \( \tau = 0 \) and 1 only. Below procedure is used for \( \tau = 2 \) that allows us to extend region of stability of the equation under consideration.

3. LINEAR VOLterra EQUATIONS WITH CONSTANT COEFFICIENTS

Let us apply the proposed procedure for the investigation of the scalar equation

\[ x_{i+1} = \sum_{j=-h}^{i} a_{i-j} x_j + \sum_{j=0}^{i} \sigma_{i-j} x_j, \quad i \in \mathbb{Z}, \]

\[ x_i = \phi_i, \quad i \in \mathbb{Z}_0. \]  

(7)

Here \( a_i \) and \( \sigma^i_j \) are some known constants.

1. Following the first step of the procedure let us represent the right-hand side of equation (7) in form (5) with

\[ \tau = 2, \quad F_1(i, x_{i-2}, x_{i-1}, x_i) = a_0x_i + a_1x_{i-1} + a_2x_{i-2} \]

(8)

\[ F_2(i, x_{-h}, \ldots, x_i) = \sum_{l=-h}^{i-3} a_{i-l} x_l \]

\[ F_3(i, x_{-h}, \ldots, x_i) = G_1(i, j, x_{j-2}, x_{j-1}, x_j) = 0 \]

\[ G_2(i, j, x_{-h}, \ldots, x_i) = \sum_{l=-h}^{j} \sigma_{j-l} x_l, \quad j = 0, \ldots, i, \quad i = 0, 1, \ldots \]

2. In this case the auxiliary equation type of (6) is

\[ y_{i+1} = a_0y_i + a_1y_{i-1} + a_2y_{i-2} \]
Introduce into consideration the vector \( y(i) = (y_{i-2}, y_{i-1}, y_i)' \) and represent equation (8) in the form

\[
y(i + 1) = Ay(i), \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_2 & a_1 & a_0 \end{pmatrix}
\]  

(9)

Consider now the matrix equation

\[
A'DA - D = -U, \quad U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  

(10)

where \( D \) is a symmetric matrix. The entries \( d_{ij} \) of the matrix \( D \) are defined by the following system of the equations:

\[
a_2^2d_{33} - d_{11} = 0
\]

(11)

\[
a_{2}d_{13} + a_1a_2d_{33} - d_{12} = 0
\]

\[
a_2d_{23} + a_0a_2d_{33} - d_{13} = 0
\]

\[
d_{11} + 2a_1d_{13} + a_1^2d_{33} - d_{22} = 0
\]

\[
d_{12} + a_0d_{13} + a_0a_1d_{33} + (a_1 - 1)d_{23} = 0
\]

\[
d_{22} + 2a_0d_{23} + (a_0^2 - 1)d_{33} = -1
\]

Solving system (11) we obtain

\[
d_{11} = a_2^2d_{33}
\]

\[
d_{12} = \frac{a_2(1 - a_1)(a_1 + a_0a_2)}{1 - a_1 - a_2(a_0 + a_2)} d_{33}
\]

\[
d_{13} = \frac{a_2(a_0 + a_1a_2)}{1 - a_1 - a_2(a_0 + a_2)} d_{33}
\]

\[
d_{22} = \left[ a_1^2 + a_2^2 + \frac{2a_1a_2(a_0 + a_1a_2)}{1 - a_1 - a_2(a_0 + a_2)} \right] d_{33}
\]

\[
d_{23} = \frac{(a_0 + a_2)(a_1 + a_0a_2)}{1 - a_1 - a_2(a_0 + a_2)} d_{33}
\]

\[
d_{33} = \left[ 1 - a_0^2 - a_1^2 - a_2^2 - 2 \frac{a_1a_2(a_0 + a_1a_2) + a_0(a_0 + a_2)(a_1 + a_0a_2)}{1 - a_1 - a_2(a_0 + a_2)} \right]^{-1}
\]

(12)

Let us suppose that the solution \( D \) of matrix equation (10) is a positive semidefinite matrix with \( d_{33} > 0 \). In this case the function \( v_i = y'(i)Dy(i) \) is Lyapunov function for
Equation (8). Really,
\[ \Delta v_i = y'(i+1)Dy(i+1) - y'(i)Dy(i) = y'(i)A'DA - D]y(i) = -y'(i)Uy(i) = -y_i^2 \]

3. Following the third step of the procedure the main part \( V_i \) of Lyapunov functional \( V_i = V_{i1} + V_{i2} \) must be chosen in the form \( V_{i1} = x'(i)Dx(i) \) where \( x(i) = (x_{i-2}, x_{i-1}, x_i)' \).

Represent equation (1) in the form
\[ x(i + 1) = Ax(i) + B(i) \] (13)
where the matrix \( A \) is defined in (9) and \( B(i) = (0, 0, b_i)' \),
\[ b_i = \sum_{l=-h}^{i-3} a_{i-l}x_l + \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j}x_l \xi_j \] (14)

Calculating \( E\Delta V_{i1} \), by virtue of Equation (13) we obtain
\[ E\Delta V_{i1} = E[x'(i+1)Dx(i+1) - x'(i)Dx(i)] = E[(Ax(i) + B(i))'D(Ax(i) + B(i)) - x'(i)Dx(i)] = E[-x^2 + 2B'(i)DAx(i) + B'(i)DB(i)] \]

Using the second and the third equations of system (11) it is easy to get
\[ B'(i)DB(i) = d_{33}b_i^2 \]
\[ B'(i)DAx(i) = \left( \frac{d_{13}}{a_2} x_i + \frac{d_{12}}{a_2} x_{i-1} + a_2 d_{33} x_{i-2} \right) b_i \]

From (12) it follows that the expressions \( d_{13}/a_2 \) and \( d_{13}/a_2 \) are definite for \( a_2 = 0 \) too. As a result, using (14) we obtain
\[ E\Delta V_{i1} = E \left[ -x_i^2 + d_{33}b_i^2 + 2 \left( \frac{d_{13}}{a_2} x_i + \frac{d_{12}}{a_2} x_{i-1} + a_2 d_{33} x_{i-2} \right) b_i \right] = -Ex_i^2 + d_{33} E b_i^2 + 2 \frac{d_{13}}{a_2} Ex_i b_i + 2 \frac{d_{12}}{a_2} Ex_{i-1} b_i + 2a_2 d_{33} Ex_{i-2} b_i = -Ex_i^2 + \sum_{k=1}^{9} I_k \]
where
\[ I_1 = d_{33} E \left( \sum_{i=-h}^{i-3} a_{i-l}x_l \right) ^2 , \quad I_2 = d_{33} E \left( \sum_{l=-h}^{i} \sum_{j=-h}^{j} \sigma_{j-l}^{i-j}x_l \xi_j \right) ^2 \]
\[ I_3 = 2d_{33} E \sum_{k=-h}^{i-3} a_{i-k}x_k \sum_{l=-h}^{i} \sum_{j=-h}^{j} \sigma_{j-l}^{i-j}x_l \xi_j \]
\[ I_4 = 2 \frac{d_{13}}{a_2} Ex_i \sum_{l=-h}^{i-3} a_{i-l}x_l , \quad I_5 = 2 \frac{d_{13}}{a_2} Ex_i \sum_{l=-h}^{i} \sum_{j=-h}^{j} \sigma_{j-l}^{i-j}x_l \xi_j \]

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\[ I_6 = 2 \frac{d_{12}}{a_2} \mathbb{E} x_{i-1} \sum_{l=-h}^{i-3} a_{i-k} x_k, \quad I_7 = 2 \frac{d_{12}}{a_2} \mathbb{E} x_{i-1} \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l} x_l \xi_j \]

\[ I_8 = 2a_2 d_{33} \mathbb{E} x_{i-2} \sum_{l=-h}^{i-3} a_{i-k} x_k, \quad I_9 = 2a_2 d_{33} \mathbb{E} x_{i-2} \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l} x_l \xi_j \]

Let us estimate the summands \( I_k, k = 1, \ldots, 9 \), using the following notation:

\[ k_m = \max(k, 0), \quad \alpha_3 = \sum_{l=3}^{\infty} |a_l| \]

\[ S_0 = \sum_{p=0}^{\infty} \left( \sum_{l=0}^{\infty} |\sigma_{l}^p| \right)^2, \quad S_k = \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} |\sigma_{j}^p|, \quad k = 1, 2, 3, 4 \quad (15) \]

For \( I_1, I_2 \) we have

\[ I_1 \leq d_{33} \sum_{l=-h}^{i-3} |a_{i-l}| \sum_{k=-h}^{i-3} |a_{i-k}| \mathbb{E} x_k^2 \leq \alpha_3 d_{33} \sum_{k=-h}^{i-3} |a_{i-k}| \mathbb{E} x_k^2 \]

\[ I_2 = d_{33} \sum_{j=0}^{i} \mathbb{E} \left( \sum_{l=-h}^{j} \sigma_{j-l} x_l \right)^2 \leq d_{33} \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l} \sum_{k=-h}^{j} |\sigma_{j-l}^p| \mathbb{E} x_k^2 \]

\[ \leq d_{33} \sum_{k=-h}^{i} \sum_{j=k}^{i} |\sigma_{j-k}^p| \sum_{l=0}^{\infty} |\sigma_{l}^p| \mathbb{E} x_k^2 \]

\[ = d_{33} \left( |\sigma_0^p| \sum_{l=0}^{\infty} |\sigma_{l}^p| \mathbb{E} x_l^2 + \sum_{p=0}^{1} |\sigma_{p}^p| \sum_{l=0}^{\infty} |\sigma_{l}^p| \mathbb{E} x_{l-1}^2 \right. \]

\[ + \sum_{p=0}^{2} |\sigma_{p}^p| \sum_{l=0}^{\infty} |\sigma_{l}^p| \mathbb{E} x_{l-2}^2 + \sum_{k=-h}^{i-3} \sum_{p=0}^{i-k} |\sigma_{p}^p| \sum_{l=0}^{\infty} |\sigma_{l}^p| \mathbb{E} x_k^2 \right) \]

Using properties of conditional expectation for \( I_3 \) we obtain

\[ |I_3| = 2d_{33} \mathbb{E} \sum_{k=-h}^{i-3} a_{i-k} x_k \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l} x_l \xi_j \]

\[ = 2d_{33} \left( \sum_{k=-h}^{i-3} a_{i-k} x_k \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{j-l} x_l \xi_j \right) \]

\[ = 2d_{33} \left( \sum_{j=0}^{i} \left( \sum_{k=-h}^{i-j} a_{i-k} x_k + \sum_{k=j+1}^{i-3} a_{i-k} x_k \right) \sum_{l=-h}^{j} \sigma_{j-l} x_l \xi_j \right) \]
\[\begin{align*}
&= 2d_3 \left| \sum_{j=0}^{i-4} \sum_{k=j+1}^{i-3} a_{i-k} x_k \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \psi_j \right| \\
&\leq d_3 \sum_{j=0}^{i-4} \sum_{k=j+1}^{i-3} \left| \sigma_{j-l}^{i-j} \sum_{l=-h}^{j} \left| a_{i-k} (E x_l^2 + E x_k^2) \right| \right| \\
&= d_3 \sum_{j=0}^{i-4} \sum_{k=j+1}^{i-3} \left| \sigma_{j-l}^{i-j} \sum_{l=-h}^{j} \left| a_{i-k} E x_l^2 + d_3 \sum_{j=0}^{i-4} \sum_{k=j+1}^{i-3} \left| \sigma_{j-l}^{i-j} \sum_{l=-h}^{j} \left| a_{i-k} E x_k^2 \right| \right| \right| \\
&\leq \alpha_3 d_3 \sum_{j=0}^{i-4} \sum_{k=j+1}^{i-3} \left| \sigma_{j-l}^{i-j} \sum_{l=-h}^{j} \left| a_{i-k} E x_k^2 \right| + d_3 \sum_{j=0}^{i-4} \sum_{k=j+1}^{i-3} \left| \sigma_{j-l}^{i-j} \sum_{l=-h}^{j} \left| a_{i-k} E x_k^2 \right| \right| \right| \\
&\leq \alpha_3 d_3 \sum_{j=0}^{i-4} \sum_{k=j+1}^{i-3} \left| \sigma_{j-l}^{i-j} \sum_{l=-h}^{j} \left| a_{i-k} E x_k^2 \right| + S_4 d_3 \sum_{k=1}^{i-3} \left| a_{i-k} E x_k^2 \right| \right|
\end{align*}\]

Similar for the summands \( I_4 - I_9 \) we get

\[\begin{align*}
|I_4| &\leq \frac{d_{13}}{a_2} \sum_{l=-h}^{i-3} \left| a_{i-l} \left( E x_l^2 + E x_k^2 \right) \right| \leq \frac{d_{13}}{a_2} \left( \alpha_3 E x_l^2 + \sum_{k=-h}^{i-3} \left| a_{i-k} E x_k^2 \right| \right) \\
|I_5| &\leq \frac{2 d_{13}}{a_2} \sum_{j=0}^{i-1} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \phi_j \leq \frac{d_{13}}{a_2} \sum_{j=0}^{i-1} \sum_{l=-h}^{j} \left| \sigma_{j-l}^{i-j} \left( E x_l^2 + E x_k^2 \right) \right| \\
&\leq \frac{d_{13}}{a_2} \left( S_4 E x_l^2 + \left| \sigma_{0}^{i-l} \right| E x_{i-1}^2 + \sum_{p=0}^{2} \left| \sigma_{2-p}^{i-l} \right| E x_{i-2}^2 + \sum_{k=-h}^{i-3} \sum_{p=4}^{i-k_m} \left| \sigma_{i-k_m}^{i-l} \right| E x_k^2 \right) \\
|I_6| &\leq \frac{d_{12}}{a_2} \sum_{l=-h}^{i-3} \left| a_{i-l} \left( E x_{i-1}^2 + E x_i^2 \right) \right| \leq \frac{d_{12}}{a_2} \left( \alpha_3 E x_{i-1}^2 + \sum_{k=-h}^{i-3} \left| a_{i-k} E x_k^2 \right| \right) \\
|I_7| &\leq \frac{2 d_{12}}{a_2} \sum_{j=0}^{i-2} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l x_j \psi_j \leq \frac{d_{12}}{a_2} \sum_{j=0}^{i-2} \sum_{l=-h}^{j} \left| \sigma_{j-l}^{i-j} \left( E x_l^2 + E x_k^2 \right) \right| \\
&\leq \frac{d_{12}}{a_2} \left( S_2 E x_{i-1}^2 + \left| \sigma_{0}^{i-l} \right| E x_{i-2}^2 + \sum_{p=2}^{i-k_m} \sum_{k=-h}^{i-3} \left| \sigma_{i-k_m}^{i-l} \right| E x_k^2 \right) \\
|I_8| &\leq \left| a_2 \right| d_{33} \sum_{l=-h}^{i-3} \left| a_{i-l} \left( E x_{i-2}^2 + E x_i^2 \right) \right| \leq \left| a_2 \right| d_{33} \left( \alpha_3 E x_{i-2}^2 + \sum_{k=-h}^{i-3} \left| a_{i-k} E x_k^2 \right| \right) \\
|I_9| &\leq \left| 2 a_2 d_{33} E x_{i-2} \sum_{j=0}^{i-3} \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \phi_j \right| \leq \left| a_2 \right| d_{33} \sum_{j=0}^{i-3} \sum_{l=-h}^{j} \left| \sigma_{j-l}^{i-j} \left( E x_{i-2}^2 + E x_i^2 \right) \right| \\
&\leq \left| a_2 \right| d_{33} \left( S_3 E x_{i-2}^2 + \sum_{k=-h}^{i-k_m} \sum_{p=3}^{i-3} \left| \sigma_{i-k_m}^{i-l} \right| E x_k^2 \right)
\end{align*}\]
As a result we obtain
\[
\mathbf{E}_t \Delta V_i \leq (\gamma_0 - 1) \mathbf{E}_t x_i^2 + \gamma_1 \mathbf{E}_t x_{i-1}^2 + \gamma_2 \mathbf{E}_t x_{i-2}^2 + \sum_{k=-h}^{i-3} P_{ik} \mathbf{E}_t x_k^2
\] \tag{16}

where
\[
\gamma_0 = d_{33} |\sigma_0^0| \sum_{l=0}^{\infty} |\sigma_i^l| + \frac{d_{13}}{a_2} (S_1 + \alpha_3)
\]
\[
\gamma_1 = d_{33} \sum_{p=0}^{1} |\sigma_i^{p-1}| \sum_{l=0}^{\infty} |\sigma_i^l| + \frac{d_{13}}{a_2} |\sigma_i^0| + \frac{d_{12}}{a_2} (S_2 + \alpha_3)
\]
\[
\gamma_2 = d_{33} \sum_{p=0}^{2} |\sigma_i^{p-2}| \sum_{l=0}^{\infty} |\sigma_i^l| + \frac{d_{13}}{a_2} \sum_{p=1}^{2} |\sigma_i^{p-1}| + \frac{d_{12}}{a_2} |\sigma_i^0| + |a_2|d_{33}(S_3 + \alpha_3)
\] \tag{17}

\[
P_{ik} = d_{33} \left(S_4 a_{i-k} + \sum_{p=0}^{i-k} |\sigma_i^{p-1} a_{i-p}| \sum_{l=0}^{\infty} |\sigma_i^l| \right)
\]
\[
+ \frac{d_{13}}{a_2} \left(|a_{i-k}| + \sum_{p=1}^{i-k} |\sigma_i^{p-1} a_{i-p}| \right) + \frac{d_{12}}{a_2} \left(|a_{i-k}| + \sum_{p=2}^{i-k} |\sigma_i^{p-1} a_{i-p}| \right)
\]
\[
+ |a_2|d_{33} \left(|a_{i-k}| + \sum_{p=3}^{i-k} |\sigma_i^{p-3} a_{i-p}| \right) + \alpha_3 d_{33} \left(|a_{i-k}| + \sum_{p=4}^{i-k} |\sigma_i^{p-1} a_{i-p}| \right)
\] \tag{18}

4. Following the fourth step of the procedure of Lyapunov functionals construction choose now the functional \( V_{2i} \) in the form
\[
V_{2i} = (\gamma_1 + \gamma_2 + \mu) x_i^2 + (\gamma_2 + \mu) x_{i-1}^2 + \sum_{k=-h}^{i-3} x_k^2 \sum_{j=0}^{\infty} P_{j+i-k}
\]
where numbers \( \gamma_1, \gamma_2 \) are defined by (17) and a positive number \( \mu \) will be chosen below.
Calculating \( \Delta V_{2i} \) we obtain
\[
\Delta V_{2i} = (\gamma_1 + \gamma_2 + \mu) x_i^2 - \gamma_1 x_i^2 - (\gamma_2 + \mu) x_{i-2}^2 - \sum_{j=3}^{i-3} P_{j+i-2j-2} x_{i-2}^2 \sum_{k=-h}^{i-3} P_{j+i-k}
\]
Using (16), for the functional \( V_i = V_{1i} + V_{2i} \) we get
\[
\mathbf{E}_t \Delta V_i \leq (\gamma_0 + \gamma_1 + \gamma_2 + \mu - 1) \mathbf{E}_t x_i^2 + \left( \sum_{j=3}^{i-3} P_{j+i-2j-2} - \mu \right) \mathbf{E}_t x_{i-2}^2
\]
Putting now
\[
\mu = \sup_{i \in \mathbb{Z}} \sum_{j=3}^{i-3} P_{j+i-2j-2}
\] \tag{19}
as a result we obtain
\[
\mathbf{E}_t \Delta V_i \leq (\gamma_0 + \gamma_1 + \gamma_2 + \mu - 1) \mathbf{E}_t x_i^2
\]
Thus, if the inequality
\[ \gamma_0 + \gamma_1 + \gamma_2 + \mu < 1 \]  \hfill (20)
holds then the constructed functional \( V' \) satisfies the conditions of Theorem 1.1 and therefore the trivial solution of equation (7) is asymptotically mean square stable.

Let us transform condition (20) to more obvious form. For this aim estimate \( \mu \). From (15) it follows that \( j + k - k_m \leq j \). Therefore, using (15) and (18), we have
\[
\sum_{j=3}^{\infty} P_{j+i-2,j-2} \leq \sum_{j=3}^{\infty} \left[ d_{33} \left( S_4 |a_j| + \sum_{p=0}^{j} |\sigma_{j-p}^p| \sum_{l=0}^{\infty} |\sigma_l^l| \right) \right]
+ \frac{d_{13}}{a_2} \left( |a_j| + \sum_{p=1}^{j} |\sigma_{j-p}^p| \right) + \frac{d_{12}}{a_2} \left( |a_j| + \sum_{p=2}^{j} |\sigma_{j-p}^p| \right)
+ |a_2|d_{33} \left( |a_j| + \sum_{p=3}^{j} |\sigma_{j-p}^p| \right) + a_3d_{33} \left( |a_j| + \sum_{p=4}^{j} |\sigma_{j-p}^p| \right)
= a_3 \left[ d_{33}(S_4 + |a_2| + a_3) + \frac{|d_{12}| + |d_{13}|}{a_2} \right]
+ d_{33} \sum_{j=3}^{\infty} \sum_{p=0}^{j} |\sigma_{j-p}^p| \sum_{l=0}^{\infty} |\sigma_l^l| + \frac{d_{13}}{a_2} \sum_{j=3}^{\infty} \sum_{p=1}^{j} |\sigma_{j-p}^p| + \frac{d_{12}}{a_2} \sum_{j=3}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^p|
+ |a_2|d_{33} \sum_{j=3}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^p| + a_3d_{33} \sum_{j=3}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^p|
\]

From here and (17) it follows that
\[
\gamma_0 + \gamma_1 + \gamma_2 + \mu \leq d_{33} |\sigma_0^0| \sum_{l=0}^{\infty} |\sigma_l^l| + \frac{d_{13}}{a_2} (S_1 + a_3)
+ d_{33} \sum_{p=0}^{1} |\sigma_{1-p}^p| \sum_{l=0}^{\infty} |\sigma_l^l| + \frac{d_{13}}{a_2} |\sigma_0^0| + \frac{d_{12}}{a_2} (S_2 + a_3)
+ d_{33} \sum_{p=0}^{2} |\sigma_{2-p}^p| \sum_{l=0}^{\infty} |\sigma_l^l| + \frac{d_{13}}{a_2} |\sigma_2^2| + \frac{d_{12}}{a_2} |\sigma_0^0| + |a_2|d_{33}(S_3 + a_3)
+ a_3 \left[ d_{33}(S_4 + |a_2| + a_3) + \frac{|d_{12}| + |d_{13}|}{a_2} \right]
+ d_{33} \sum_{j=3}^{\infty} \sum_{p=0}^{j} |\sigma_{j-p}^p| \sum_{l=0}^{\infty} |\sigma_l^l| + \frac{d_{13}}{a_2} \sum_{j=3}^{\infty} \sum_{p=1}^{j} |\sigma_{j-p}^p| + \frac{d_{12}}{a_2} \sum_{j=3}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^p|
\]
\[ + \frac{d_{12}}{a_2} \sum_{j=3}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^{p}| + |a_2|d_{33} \sum_{j=3}^{\infty} \sum_{p=3}^{j} |\sigma_{j-p}^{p}| \]

\[ + \alpha_3d_{33} \sum_{j=4}^{\infty} \sum_{p=4}^{j} |\sigma_{j-p}^{p}| \]

\[ = \frac{d_{13}}{a_2} (S_1 + \alpha_3) + \frac{d_{12}}{a_2} (S_2 + \alpha_3) + |a_2|d_{33}(S_3 + \alpha_3) \]

\[ + \alpha_3 \left[ d_{33}(S_4 + |a_2| + \alpha_3) + \frac{|d_{12}| + |d_{13}|}{|a_2|} \right] \]

\[ + d_{33} \sum_{j=0}^{\infty} \sum_{p=0}^{j} |\sigma_{j-p}^{p}| \sum_{l=0}^{\infty} |\sigma_{j-l}^{p}| + \frac{d_{13}}{|a_2|} \sum_{j=1}^{\infty} \sum_{p=1}^{j} |\sigma_{j-p}^{p}| \]

\[ + \frac{d_{12}}{a_2} \sum_{j=2}^{\infty} \sum_{p=2}^{j} |\sigma_{j-p}^{p}| + |a_2|d_{33} \sum_{j=3}^{\infty} \sum_{p=3}^{j} |\sigma_{j-p}^{p}| \]

Using notation (15) we have

\[ \sum_{j=0}^{\infty} \sum_{p=0}^{j} |\sigma_{j-p}^{p}| \sum_{l=0}^{\infty} |\sigma_{j-l}^{p}| = \sum_{p=0}^{\infty} \sum_{j=p}^{\infty} |\sigma_{j-p}^{p}| \sum_{l=0}^{\infty} |\sigma_{j-l}^{p}| = \sum_{p=0}^{\infty} \left( \sum_{l=0}^{\infty} |\sigma_{j-l}^{p}| \right)^2 = S_0 \]

\[ \sum_{j=k}^{\infty} \sum_{p=k}^{j} |\sigma_{j-p}^{p}| = \sum_{p=k}^{\infty} \sum_{j=p}^{\infty} |\sigma_{j-p}^{p}| = \sum_{p=k}^{\infty} \sum_{l=0}^{\infty} |\sigma_{j-l}^{p}| = S_k, \quad k = 1, 2, 3, 4 \]

Therefore, we obtain

\[ \gamma_0 + \gamma_1 + \gamma_2 + \mu \leq \frac{d_{13}}{a_2} (S_1 + \alpha_3) + \frac{d_{12}}{a_2} (S_2 + \alpha_3) + |a_2|d_{33}(S_3 + \alpha_3) \]

\[ + \alpha_3 \left[ d_{33}(S_4 + |a_2| + \alpha_3) + \frac{|d_{12}| + |d_{13}|}{|a_2|} \right] \]

\[ + d_{33}S_0 + \frac{d_{13}}{a_2} S_1 + \frac{d_{12}}{a_2} S_2 + |a_2|d_{33}S_3 + \alpha_3d_{33}S_4 \]

\[ = 2\frac{d_{13}}{a_2} (S_1 + \alpha_3) + 2\frac{d_{12}}{a_2} (S_2 + \alpha_3) \]

\[ + d_{33}[S_0 + 2|a_2|(S_3 + \alpha_3) + 2\alpha_3S_4 + \alpha_3^2] \]
Using representation (12) for $d_{12}$, $d_{13}$, we have

$$
\gamma_0 + \gamma_1 + \gamma_2 + \mu \leq 2d_{33}\left[ \frac{|a_0 + a_1a_2|(S_1 + x_3) + |(1 - a_1)(a_1 + a_0a_2)|(S_2 + x_3)}{|1 - a_1 - a_2(a_0 + a_2)} \right]
+ d_{33}[S_0 + 2|a_2|(S_3 + x_3) + 2x_3S_4 + x_3^2]
$$

Thus, using (12), we obtain that condition (20) follows from inequality:

$$
2\left[ \frac{|a_0 + a_1a_2|(S_1 + x_3) + |(1 - a_1)(a_1 + a_0a_2)|(S_2 + x_3)}{|1 - a_1 - a_2(a_0 + a_2)} \right]
+ S_0 + 2|a_2|(S_3 + x_3) + 2x_3S_4 + x_3^2 < d_{33}^{-1}
$$

$$
= 1 - a_0^2 - a_1^2 - a_2^2 - 2a_0a_1a_2 - \frac{2(a_0 + a_2)(a_0 + a_1a_2)(a_1 + a_0a_2)}{1 - a_1 - a_0a_2 - a_2^2}
$$

(21)

So, the following theorem is proven.

**Theorem 3.1**

If the matrix $D$ with the entries defined by (12) is a positive semidefinite one and the inequality (21) holds then the trivial solution of equation (7) is asymptotically mean square stable.

### 4. EXAMPLES

**Example 4.1**

Consider the scalar equation

$$
x_{i+1} = \sum_{j=-h}^{i} a_{i-j}x_j + \sigma x_{i-m} \xi_i, \quad m \geq 0
$$

(22)

From (15) for Equation (22) it follows that:

$$
S_0 = \sigma^2, \quad S_k = 0, \quad k > 0
$$

Then condition (21) takes the form

$$
x_3^2 + 2x_3 \left[ a_2 + \frac{|a_0 + a_1a_2| + |(1 - a_1)(a_1 + a_0a_2)|}{|1 - a_1 - a_2(a_0 + a_2)|} \right] + \sigma^2 < d_{33}^{-1}
$$

(23)

In particular, for the equation

$$
x_{i+1} = a_0x_i + a_1x_{i-1} + a_2x_{i-2} + \sigma x_{i-m} \xi_i
$$

(24)

we have $x_3 = 0$ and from condition (23) it follows:

$$
\sigma^2 < d_{33} < 1
$$

(25)

As it is shown in Reference [2] if the matrix $D$ with the entries defined by (12) is a positive semidefinite one with $d_{33} > 0$ then inequality (25) is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (24).
Example 4.2
Consider the scalar equation

\[ x_{i+1} = ax_i + \sum_{j=1}^{i+h} b'x_{i-j} + \sigma x_{i-m}\xi_i \]

|b| < 1, \quad m \geq 0 \quad (26)

Using the procedure of Lyapunov functional construction described above for \( \tau = 0 \) the following condition for asymptotic mean square stability of the trivial solution of equation (26) was obtained [1]

\[ |a| + \frac{|b|}{1 - |b|} < \sqrt{1 - \sigma^2} \quad (27) \]

Using this procedure for \( \tau = 1 \) in Reference [1] was obtained another stability condition, which can be written in the form

\[ |a| + b < 1 \]

\[ \frac{b^2}{1 - |b|} \left[ \frac{|b(2 - |b|) + 2|a|}{1 - |b|} + \frac{2\sigma^2}{1 - b} \right] + a^2 < 1 - b^2 - a^2 - \frac{1 + b}{1 - b} \quad (28) \]

From conditions (12), (15), (21), which were obtained using the procedure for \( \tau = 2 \), it follows that if the symmetric matrix \( D \) with the entries:

\[ d_{11} = b^4 d_{33}, \quad d_{12} = \frac{b^3(1 - b)(1 + ab)}{1 - b - b^2(a + b^2)} d_{33} \]

\[ d_{13} = \frac{b^2(a + b^3)}{1 - b - b^2(a + b^2)} d_{33}, \quad d_{23} = \frac{b(a + b^2)(1 + ab)}{1 - b - b^2(a + b^2)} d_{33} \]

\[ d_{22} = b^2 \left[ 1 + b^2 + \frac{2b(a + b^3)}{1 - b - b^2(a + b^2)} \right] d_{33} \]

\[ d_{33} = \left[ 1 - a^2 - b^2 - b^4 - 2b^2(a + b^3) + a(a + b^2)(1 + ab) \right]^{-1} \]

is a positive definite one and the inequality

\[ \frac{|b|^6}{(1 - |b|)^2} + \frac{2|b|^3}{1 - |b|} \left[ b^2 + \frac{|a + b^3| + (1 - b)b(1 + ab)}{|1 - b - b^2(a + b^2)|} \right] + \sigma^2 < 1 - a^2 - b^2 - b^4 - 2b^2(a + b^3) + a(a + b^2)(1 + ab) \]

\[ \frac{1 - b - b^2(a + b^2)}{1 - b - b^2(a + b^2)} \quad (29) \]

holds then the trivial solution of equation (25) is asymptotically mean square stable.

In Figure 1, stability regions are shown, given by conditions (27) (the curve number 1), (28) (the curve number 2) and (29) (the curve number 3) for \( \sigma^2 = 0 \). In Figure 2 we can see the similar picture for \( \sigma^2 = 0.3 \) and in Figure 3 the similar picture is shown (in doubled scale) for \( \sigma^2 = 0.7 \).
Figure 1.

Figure 2.

Figure 3.
Remark 4.1
As it is shown in Figure 1 (and naturally it can be shown analytically) if \( \sigma = 0 \) then for \( b \geq 0 \) condition (27) coincides with condition (28) and for \( a \geq 0, b \geq 0 \) condition (29) coincides with conditions (27) and (28).

Remark 4.2
It is easy to see that stability region \( Q_t \) obtained for equation (26) expands if \( \tau \) increases, i.e., \( Q_0 < Q_1 < Q_2 \). So, if we want to get greater stability region we can use the procedure of Lyapunov functionals construction for \( \tau = 3, 4, \) etc.

REFERENCES