Definition 1. A matrix $A$ is in reduced row echelon form (rref) if

1. Every zero row in $A$ is below every nonzero row.
2. The leading term in any nonzero row in $A$ is strictly to the right of all the leading terms in the rows above it.
3. The leading term in a nonzero row in $A$ is a 1, and is the only nonzero entry in its column.

A system of linear equations $Ax = b$ is in reduced row echelon form if and only if $A$ is.

Definition 2. Let $A$ be an $m \times n$ matrix.

1. An $n \times m$ matrix $B$ such that $BA = I_n$ is called a left inverse for $A$.
2. An $n \times m$ matrix $C$ such that $AC = I_m$ is called a right inverse for $A$.
3. An $n \times m$ matrix $D$ which is both a left inverse and a right inverse for $A$ is called an inverse for $A$.

If $A$ has a left inverse or right inverse, $A$ is called left-invertible or right-invertible respectively. $A$ is called invertible if it is both left-invertible and right-invertible.

Notice that, on the face of it, a matrix may be invertible but have no inverse. The following result should correct our doubt about that.

Fact 1. Suppose $A$ is invertible, and $B$ and $C$ are a left inverse and a right inverse for $A$, respectively. Then $B = C$, and hence $A$ has an inverse.

Proof. We have

$$B = BI = B(AC) = (BA)C = IC = C$$

□

We also get a uniqueness theorem out of this, which will allow us to make the following definition.

Corollary 1. Suppose $A$ is invertible. Then there is exactly one inverse for $A$. 
Proof. If \(B\) and \(C\) are inverses for \(A\), then in particular, \(B\) is a left inverse and \(C\) a right inverse. Then \(B = C\) by the above fact. \(\Box\)

**Definition 3.** If \(A\) is an invertible matrix, then \(A^{-1}\) is the unique inverse of \(A\).

**Theorem 1.** Let \(A\) be an \(m \times n\) matrix. Then the following are equivalent.

1. For any choice of constants \(b_1, \ldots, b_m\), the system of linear equations
   \[
   a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
   a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\
   \vdots \\
   a_{m1}x_1 + \cdots + a_{mn}x_n = b_m
   \]
   has at least one solution.
2. For every \(b \in \mathbb{R}^m\), there is some \(s \in \mathbb{R}^n\) such that \(As = b\).
3. \(T_A : \mathbb{R}^n \to \mathbb{R}^m\) is surjective.
4. \(A\) is right-invertible.

Proof. The equivalence of (1), (2), and (3) is just a recapitulation of the definitions in play. The real work is in proving they’re equivalent to (4).

I’ll prove (2) is equivalent to (4). First, assume (2); we’ll prove (4). Consider the standard basis vectors in \(\mathbb{R}^m\):

\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}
\]

By our assumption, there are vectors \(s_1, \ldots, s_m \in \mathbb{R}^n\) such that \(Ae_1 = s_1, \ldots, Aes_m = s_m\). Now let \(B\) be the matrix with columns \(s_1, \ldots, s_m\), in that order. Then \(B\) is \(n \times m\), and for any \(i, j\), \(B_{ij} = (s_j)_i\), the \(i\)th entry of column vector \(s_j\). Then for all \(i, j \leq m\), we have

\[
(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ik}(s_j)_k = (As_j)_i = (e_j)_i = I_{ij}
\]

so \(AB = I\).

Now assume (4), and let \(B\) be a right inverse for \(A\). Let \(b \in \mathbb{R}^m\) be given. Then

\[
A(Bb) = (AB)b = Ib = b
\]

and so there is some \(s \in \mathbb{R}^n\) such that \(As = b\), namely \(s = Bb\). \(\Box\)