MONOTONE PATHS

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Joint work with Mikhail Lavrov
Theorem (Erdős-Szekeres 1935)

Every permutation of \( \{1, \ldots, n\} \) has a monotone subsequence of length about \( \sqrt{n} \).
**Monotone sequences**

**Theorem (Erdős-Szekeres 1935)**
Every permutation of \(\{1, \ldots, n\}\) has a monotone subsequence of length about \(\sqrt{n}\).

**Example**

\[
1 \hspace{1em} 5 \hspace{1em} 2 \hspace{1em} 7 \hspace{1em} 3 \hspace{1em} 6 \hspace{1em} 4
\]
**Theorem (Erdős-Szekeres 1935)**

Every permutation of \( \{1, \ldots, n\} \) has a monotone subsequence of length about \( \sqrt{n} \).

**Example**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>5</th>
<th>2</th>
<th>7</th>
<th>3</th>
<th>6</th>
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</tr>
</thead>
<tbody>
<tr>
<td>inc.</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
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<tr>
<td>dec.</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Proof.** Under each number, write lengths of longest increasing and decreasing subsequences ending there.
**Monotone walks: lower bound**

**Question (Chvátal-Komlos 1971)**

If edges of $K_n$ are ordered from $1 \ldots \binom{n}{2}$, is there always a long monotone walk?
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Every edge-ordering of $K_n$ has an increasing walk of length $n - 1$. 
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**Proof.**

![Diagram showing a walk through vertices 1, 2, 3, and 4, illustrating the increasing walk of length $n - 1$.]
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Theorem (Graham-Kleitman 1973)

There is an edge-ordering of $K_n$ in which the longest monotone walk has length $n - 1$, for all $n \notin \{3, 5\}$. 

Proof (for even $n$). Edges of $K_n$ can be partitioned into perfect matchings. Assign a batch of consecutive labels to each matching.
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A *path* in a graph is a *self-avoiding walk*, which never visits the same vertex twice.
**Definition**

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**Self-avoiding walks are more complicated**

- Easy poly-time algorithm to find longest increasing walk.
DEFINITION

A path in a graph is a self-avoiding walk, which never visits the same vertex twice.

SELF-AVOIDING WALKS ARE MORE COMPLICATED

- Easy poly-time algorithm to find longest increasing walk.
- In probability: self-avoiding random walk proven sub-ballistic only in 2012 by Duminil-Copin and Hammond.
Monotone paths

Theorem (Graham-Kleitman 1973)

Every edge-ordering of $K_n$ has an increasing path of length $\sqrt{n} - 1$.

Proof.

Employ walkers again. When edge called, if a walker would revisit a vertex, neither walker moves. Suppose all walkers take $\leq k$ steps. At most $kn^2$ edges are walked. Each walker refuses at most $(k + 1)/2 - k = (k^2)/2$ edges. ($n^2$ walked + refused $\leq kn^2 + (k^2)n^2$)

Theorem (Calderbank-Chung-Sturtevant 1984)

There is an edge-ordering of $K_n$ in which the longest increasing path has length $(1/2 - o(1))n$.
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$$\binom{n}{2} = \text{walked} + \text{refused} \leq \frac{kn}{2} + \binom{k}{2}n = \frac{k^2 n}{2}$$
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There is an edge-ordering of $K_n$ in which the longest increasing path has length $(\frac{1}{2} - o(1))n$. 
Model

Sample uniformly random ordering of $\binom{n}{2}$ edges.
Random ordering

Model

Sample uniformly random ordering of \( \binom{n}{2} \) edges. Equiv: assign independent Unif[0, 1] random real to each edge.

Proof sketch.
Start at arbitrary vertex, expose labels of incident edges. Smallest incident label is min of \( n - 1 \) Uniforms, so expectation is \( \frac{1}{n} \).

Take that edge, then expose labels of edges to \( n - 2 \) remaining vertices. Smallest increment is min of \( n - 2 \) Unifs, so expectation \( \frac{1}{n - 1} \).

Sum \( \frac{1}{n} + \frac{1}{n - 1} + \cdots + \frac{1}{cn} = 1 \) when \( \log \frac{1}{c} = 1 \).
**Random ordering**

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**Observation**
A random edge-ordering has an increasing path of length at least $(1 - \frac{1}{e})n$ a.a.s.
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- Start at arbitrary vertex, expose labels of incident edges.
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- Take that edge, then expose labels of edges to $n - 2$ remaining vertices.
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**Random Ordering**

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- Take that edge, then expose labels of edges to $n - 2$ remaining vertices.
- Smallest increment is min of $n - 2$ Unifs, so expectation $\frac{1}{n-1}$.
- Sum $\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{cn} = 1$ when $\log \frac{1}{c} = 1$. □
Trivial bound

A.a.s., a random edge-ordering does not have a Hamiltonian increasing path.

Proof. (first moment method) For a given Hamiltonian path, it is increasing with probability $\frac{1}{(n-1)!}$. Number of Hamiltonian paths is $n!$. Expected number of increasing Hamiltonian paths is $n!$. ...
A.a.s., a random edge-ordering does not have a Hamiltonian increasing path.

**Proof.** (first moment method)

- For a given Hamiltonian path, it is increasing with probability \( \frac{1}{(n-1)!} \).
**Random ordering: upper bound**

**Trivial bound**

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- Number of Hamiltonian paths is \( n! \).
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- For a given Hamiltonian path, it is increasing with probability \( \frac{1}{(n-1)!} \).
- Number of Hamiltonian paths is \( n! \).
- Expected number of increasing Hamiltonian paths is \( n \ldots \).
In Erdős-Rényi

**First moment insufficient**

$G_{n,p}$ has $n$ Hamiltonian paths on expectation when $n!p^{n-1} \sim n$, i.e., when $p \sim \frac{e}{n}$. 

**Theorem (Bollobás)**

A.a.s., random graph process gets Hamiltonian cycle at moment that all vertices have degree $\geq 2$, which is at $p \sim \log n + \log \log n + \omega n$.

**Theorem (Glebov-Krivelevich 2013)**

At hitting time, number of Hamiltonian cycles jumps from 0 to \((1 + o(1)) \log n e^{\log n} \) a.a.s.
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At hitting time, number of Hamiltonian cycles jumps from 0 to $\left(1 + o\left(1\right)\right) \log n e^n$ a.a.s.
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At hitting time, number of Hamiltonian cycles jumps from 0 to $[\left(1 + o(1)\right)\frac{\log n}{e}]^n$ a.a.s.
A random edge-ordering has an increasing Hamiltonian path with probability at least $\frac{1}{e}$.
Theorem (Lavrov, L.)

A random edge-ordering has an increasing Hamiltonian path with probability at least $\frac{1}{e}$.

Recall: greedy algorithm found increasing path of length $(1 - \frac{1}{e})n \approx 0.63n$ in a random edge-ordering, but was analyzable.
Long increasing paths

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With backtracking, $k$-greedy algorithm finds an increasing path of length $0.85n$ a.a.s. in a random edge-ordering.
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**Theorem (Lavrov, L.)**

With backtracking, $k$-greedy algorithm finds an increasing path of length $0.85n$ a.a.s. in a random edge-ordering.

**Conjecture (Lavrov, L.)**

A random edge-ordering has an increasing Hamiltonian path a.a.s.
**Theorem (Chebyshev)**

\[ P[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{Var}[X]}{t^2} \]
**Second moment method**

**Theorem (Chebyshev)**

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\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{Var}[X]}{t^2}
\]

**Theorem (Lavrov, L.)**

Let \( X \) be the number of Hamiltonian increasing paths. Then \( \mathbb{E}[X^2] \sim en^2 \).
**Theorem (Chebyshev)**

\[ \mathbb{P} \left[ |X - \mathbb{E}[X]| \geq t \right] \leq \frac{\text{Var}[X]}{t^2} \]

**Theorem (Lavrov, L.)**

Let \( X \) be the number of Hamiltonian increasing paths. Then

\[ \mathbb{E}[X^2] \sim en^2. \]

**Theorem (Paley-Zygmund)**

For nonnegative random variables \( X \),

\[ \mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \]
Profiles

**Calculation**

Let $X = I_1 + \cdots + I_n!$, a sum with one indicator random variable per potential Hamiltonian increasing path.

$$
\mathbb{E} \left[ X^2 \right] = \sum_{j,k} \mathbb{E} [I_j I_k]
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\]

**Simplest profile:** \( P, Q \) edge-disjoint

\[
P
\]
\[
Q
\]
Profiles

Calculation

Let $X = I_1 + \cdots + I_n!$, a sum with one indicator random variable per potential Hamiltonian increasing path.

$$E \left[ X^2 \right] = \sum_{j,k} E \left[ I_j I_k \right]$$

$$= \sum_{P,Q} \mathbb{P} \left[ \text{both } P \text{ and } Q \text{ increasing} \right]$$

Simplest profile: $P$, $Q$ edge-disjoint

Given $P$ and $Q$, $\mathbb{P} = \frac{1}{(n-1)!} \cdot \frac{1}{(n-1)!}$
Let $X = I_1 + \cdots + I_{n!}$, a sum with one indicator random variable per potential Hamiltonian increasing path.

$$\mathbb{E} \left[ X^2 \right] = \sum_{j,k} \mathbb{E} [I_j I_k]$$

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**Simplest profile:** $P$, $Q$ edge-disjoint

$P$  
\[-------------------\]  
$Q$

- Given $P$ and $Q$, $\mathbb{P} = \frac{1}{(n-1)!} \cdot \frac{1}{(n-1)!}$
- Number of $(P, Q)$ embeddings: $n!n!$
Let $X = I_1 + \cdots + I_n!$, a sum with one indicator random variable per potential Hamiltonian increasing path.

\[
\mathbb{E}[X^2] = \sum_{j,k} \mathbb{E}[I_j I_k] = \sum_{P,Q} \mathbb{P} \left[ \text{both } P \text{ and } Q \text{ increasing} \right]
\]

**Simplest profile:** $P$, $Q$ edge-disjoint

\begin{itemize}
  \item Given $P$ and $Q$, $\mathbb{P} = \frac{1}{(n-1)!} \cdot \frac{1}{(n-1)!}$
  \item Number of $(P, Q)$ embeddings: $n!n$
  \item Total contribution of profile: $n^2$
\end{itemize}
Let $X = I_1 + \cdots + I_{n!}$, a sum with one indicator random variable per potential Hamiltonian increasing path.

$$\mathbb{E} \left[ X^2 \right] = \sum_{j,k} \mathbb{E} \left[ I_j I_k \right]$$

$$= \sum_{P,Q} \mathbb{P} \left[ \text{both } P \text{ and } Q \text{ increasing} \right]$$

**Simplest profile:** $P$, $Q$ edge-disjoint

- Given $P$ and $Q$, $\mathbb{P} = \frac{1}{(n-1)!} \cdot \frac{1}{(n-1)!}$
- Number of $(P, Q)$ embeddings: $n!n! \frac{1}{e^2}$
- Total contribution of profile: $n^2 \frac{1}{e^2}$
Another easy profile

![Diagram with labels a, c, a', b, b']

Probability

Total number of edge labels: \(a + b + c + a' + b'\).

Lowest \(a + b\) of them must be in left branches. They can be split into top-left and bottom-left in \((a + b)\) ways.

Highest \(a' + b'\) labels can split into top-right and bottom-right in \((a' + b')\) ways, so profile probability is \((a + b)(a' + b')(a + b + c + a' + b')\)!
Another easy profile

\[ a \quad c \quad a' \]

\[ b \quad b' \]

**Probability**

- Total number of edge labels: \( a + b + c + a' + b' \).
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![Diagram](image)

**Probability**

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\[
\frac{\binom{a+b}{a} \binom{a'+b'}{a'}}{(a + b + c + a' + b')!}
\]
**Bigger profile**

\[
\text{Probability} = (a_1 + b_1)(a_2 + b_2)(a_3 + b_3)(a_4 + b_4)
\]

\[
\sum a_i + \sum b_i + \sum c_i
\]

Number of embeddings

Embed top path: \( n! \)

Bottom path has \((c_1 + 1) + (c_2 + 1) + (c_3 + 1)\) vertices already fixed.

Remaining vertices can be embedded in \((n - c_1 - c_2 - c_3 - 3)! \cdot e^{-2}\) ways.
Probability

\[
\frac{(a_1+b_1) \binom{a_1}{a_1} (a_2+b_2) \binom{a_2}{a_2} (a_3+b_3) \binom{a_3}{a_3} (a_4+b_4) \binom{a_4}{a_4}}{(\sum a_i + \sum b_i + \sum c_i)!}
\]
Probability

\[ \frac{(a_1 + b_1) (a_2 + b_2) (a_3 + b_3) (a_4 + b_4)}{(\sum a_i + \sum b_i + \sum c_i)!} \]

Number of Embeddings

- Embed top path: \( n! \)
**Probability**

\[
\frac{(a_1+b_1)}{a_1} \frac{(a_2+b_2)}{a_2} \frac{(a_3+b_3)}{a_3} \frac{(a_4+b_4)}{a_4} \frac{(\sum a_i + \sum b_i + \sum c_i)!}{(\sum a_i + \sum b_i + \sum c_i)!}
\]

**Number of Embeddings**

- Embed top path: \(n!\)
- Bottom path has \((c_1 + 1) + (c_2 + 1) + (c_3 + 1)\) vertices already fixed.
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**Number of Embeddings**

- Embed top path: \( n! \)
- Bottom path has \((c_1 + 1) + (c_2 + 1) + (c_3 + 1)\) vertices already fixed.
- Remaining vertices can be embedded in \((n - c_1 - c_2 - c_3 - 3)! \cdot e^{-2}\) ways.
Care required

When a common segment has length 1, i.e., some $c_i = 1$, the single common edge can also be traversed backwards.
**GENERAL PROFILE**

![Diagram]

**CARE REQUIRED**

When a common segment has length 1, i.e., some $c_i = 1$, the single common edge can also be traversed backwards.

**DOUBLING FACTOR**

- Probability is still

$$\frac{(a_1+b_1) \cdot (a_2+b_2) \cdot (a_3+b_3) \cdot (a_4+b_4)}{(\sum a_i + \sum b_i + \sum c_i)!}$$

- Number of embeddings is still $n!(n - c_1 - c_2 - c_3 - 3)! \cdot e^{-2}$. 
When a common segment has length 1, i.e., some $c_i = 1$, the single common edge can also be traversed backwards.

**Doubling factor**

- Probability is still

\[
\frac{(a_1 + b_1) (a_2 + b_2) (a_3 + b_3) (a_4 + b_4)}{(\sum a_i + \sum b_i + \sum c_i)!}
\]

- Number of embeddings is still $n!(n - c_1 - c_2 - c_3 - 3)! \cdot e^{-2}$.
- We pick up a factor of 2 for each $c_i = 1$. 

Therefore, second moment of number of Hamilton increasing paths is \[ \mathbb{E}[X^2] = \]

\[
\sum_{a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots} n! \left[ n - \sum (c_i + 1) \right]! e^{-2} \cdot \frac{\prod (a_i + b_i)}{[\sum a_i + \sum b_i + \sum c_i]!} \cdot 2^{\# \{i: c_i = 1\}}
\]
Therefore, second moment of number of Hamilton increasing paths is

\[\mathbb{E} [X^2] = \sum_{a_1, a_2, \ldots} n! \left[ n - \sum (c_i + 1) \right]! e^{-2} \cdot \frac{\prod (a_i + b_i)}{[\sum a_i + \sum b_i + \sum c_i]!} \cdot 2^\# \{i: c_i = 1\} \]

which, after some work, turns out to be \((1 + o(1))en^2\).
Cost of Greed

**Greedy Algorithm**
Always pick edge with smallest increment to a new vertex.

**Potential Gain**
Consider the following greedy outcome:
Greedy algorithm, with temptation

Greedy algorithm

Let $k$ be a constant, say 5.
Greedy Algorithm, with Temptation

**Greedy Algorithm**

- Let $k$ be a constant, say 5.
- When extending path, do not immediately pick smallest increment.
- Reveal *next-labeled edge to new vertex* which is incident to end of path.
Greedy algorithm, with temptation

Let $k$ be a constant, say 5.

When extending path, do not immediately pick smallest increment.

Reveal next-labeled edge to new vertex which is incident to end of path.

Reveal next edge incident to exploration tree at end of path.
Greedy algorithm, with temptation

Greedy algorithm

- Let $k$ be a constant, say 5.
- When extending path, do not immediately pick smallest increment.
- Reveal next-labeled edge to new vertex which is incident to end of path.
- Reveal next edge incident to exploration tree at end of path.
- Repeat until exploration tree has $k$ edges.
Greedy algorithm, with temptation

**Greedy algorithm**

- Let \( k \) be a constant, say 5.
- When extending path, do not immediately pick smallest increment.
- Reveal next-labeled edge to new vertex which is incident to end of path.
- Reveal next edge incident to exploration tree at end of path.
- Repeat until exploration tree has \( k \) edges.
- Extend path to earliest subtree.
GREEDY ALGORITHM, WITH TEMPTATION

**Greedy Algorithm**

- Let $k$ be a constant, say 5.
- When extending path, do not immediately pick smallest increment.
- Reveal *next-labeled edge to new vertex* which is incident to end of path.
- Reveal *next edge* incident to exploration tree at end of path.
- Repeat until exploration tree has $k$ edges.
- Extend path to earliest subtree.
- Replace exploration tree by that subtree, and repeat.
### k-Greedy Algorithm

- Let $k$ be a constant, say 5.
- When extending path, do not immediately pick smallest increment.
- Reveal **next-labeled edge to new vertex** which is incident to end of path.
- Reveal **next edge** incident to exploration tree at end of path.
- Repeat until exploration tree has $k$ edges.
- Extend path to **largest** subtree.
- Replace exploration tree by that subtree, and repeat.
Analysis of $k$-greedy

**Time to grow path from $\ell \to \ell + 1**

- Suppose exploration tree has $t$ vertices, and path has length $\ell$. 
**Analysis of \( k \)-greedy**

**Time to grow path from \( \ell \to \ell + 1 \)**

- Suppose exploration tree has \( t \) vertices, and path has length \( \ell \).
- To grow exploration tree by 1 vertex, increment is min of \( t(n - \ell) \) Uniforms, so typically \( \frac{1}{t(n - \ell)} \).

(Sanity check: in Greedy, \( t = k = 1 \).)
Analysis of $k$-greedy

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- To grow exploration tree to $k$ edges, total increment is typically

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\frac{1}{n-\ell} \left( \frac{1}{t} + \frac{1}{t+1} + \cdots + \frac{1}{k} \right).
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**Typical time to grow path from $0 \to \ell$**

\[
\left( \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-\ell} \right) \cdot \left( \text{typical } \frac{1}{t} + \cdots + \frac{1}{k} \right).
\]
**Observation**

If one watches subtrees of children of root, they grow according to the Chinese Restaurant Process.
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**Chinese Restaurant Process**
When \( n \)-th person enters restaurant:
- Start new table with probability \( \frac{1}{n} \).
- Join existing table with probability proportional to size.
Typical residual exploration tree

**Observation**

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**Chinese Restaurant Process**

When $n$-th person enters restaurant:

- Start new table with probability $\frac{1}{n}$.
- Join existing table with probability proportional to size.

**Golomb-Dickman constant**

If $T_k$ is largest table after $k$ people, then

$$\mathbb{E} \left[ \frac{T_k}{k} \right] \to 0.6243$$
**Calculation for $k$-greedy**

**Typical time to grow path from $0 \rightarrow \ell$**

$$\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-\ell}\right) \cdot \left(\text{typical } \frac{1}{T_k} + \cdots + \frac{1}{k}\right).$$
**Calculation for $k$-greedy**

**Typical time to grow path from $0 \to \ell$**

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**Typical factor**

- For large (but constant $k$):

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\frac{1}{T_k} + \cdots + \frac{1}{k} \approx \log \frac{k}{T_k}
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Calculation for $k$-greedy

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- As $k$ grows, factor decreases; for $k = 100$, factor is about 0.5219.
**Typical time to grow path from $0 \to \ell$**

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- Typical length is when

\[
\log \frac{n}{n - \ell} = \frac{1}{0.5219} \quad \Rightarrow \quad \ell = (1 - e^{-1/0.5219})n.
\]
Theorem (Lavrov, L.)

- A random edge-ordering has an increasing Hamiltonian path with probability at least $\frac{1}{e}$.
- With backtracking, $k$-greedy algorithm finds an increasing path of length $0.85n$ a.a.s. in a random edge-ordering.
- Let $X$ be the number of Hamiltonian increasing paths. Then $E[X^2] \sim en^2$.

Conjecture (Lavrov, L.)

A random edge-ordering has an increasing Hamiltonian path a.a.s.