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Erdős-Rényi $G_{n,p}$: edges appear independently with probability $p$. 

$\Pr(G_{n,p} \text{ is Hamiltonian}) \approx \frac{\sqrt{\ln n}{n^{1/2}}}{n^{1/2}}$. 

Robinson, Wormald 

$G_{3\text{-reg}}$ is Hamiltonian whp. 

Bohman, Frieze 

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Definition

$G_{n,p}$ is Hamiltonian with high probability (whp) if the probability of it not being Hamiltonian tends to 0 as $n$ tends to infinity.

($\omega(n)$ denotes an infinitesimal function of $n$.)
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**Definition (3-uniform hypergraph)**

\( H_{n,p;3} \): each triple appears independently with probability \( p \).
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![Diagram of a cycle graph](image-url)
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**Tight H-cycle**

**Loose H-cycle**

(Frieze) $H_{n,p;3}$ has loose H-cycle whp if $p > K \frac{\log n}{n^2}$, $4 \mid n$.

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Rainbow Hamilton cycles

**Question**

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Does $G_{n,p}$ have a rainbow Hamilton cycle if edges are randomly colored from $\kappa$ colors?

**Observations**

- Must have $p > \frac{\log n + \log \log n + \omega(n)}{n}$ with $\omega(n) \to \infty$.
- Must have $\kappa \geq n$. 

(Cooper, Frieze) True if $p = 20 \log n/n$ and $\kappa = 20 n$.

(Janson, Wormald) True if $G_{2r}$-reg is randomly colored with each of $\kappa = n$ colors appearing exactly $r \geq 4$ times.
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Loose vs. rainbow H-cycles

- Connect 3-uniform hypergraphs to colored graphs
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Hypergraph (bisected vertex set)
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Hypergraph (bisected vertex set)  Auxiliary graph
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- Connect 3-uniform hypergraphs (loose Hamiltonicity) to colored graphs (rainbow Hamilton cycles).

Hypergraph (bisected vertex set)  Auxiliary graph
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Connect 3-uniform hypergraphs (loose Hamiltonicity) to colored graphs (rainbow Hamilton cycles).

Frieze applied Johansson-Kahn-Vu to find perfect matchings.

Apply Janson-Wormald to find rainbow H-cycle in randomly colored random regular graph.
Theorem (Frieze, L.)

For any fixed $\epsilon > 0$, if $p = \frac{(1+\epsilon)\log n}{n}$, then $G_{n,p}$ contains a rainbow Hamilton cycle \textbf{whp} when its edges are randomly colored from $\kappa = (1 + \epsilon)n$ colors.

Remarks:

- Asymptotically best possible, both in terms of $p$ and $\kappa$.
- Still holds when $\epsilon$ tends (slowly) to zero.
Observation

If $p = \frac{(1+\epsilon) \log n}{n}$, then almost all vertices have degree $\geq \frac{1}{10} \log n$. 
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- Degree of fixed vertex is Bin \([n - 1, p]\); expectation \( \sim \log n \)
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- Typically, all but \( < \sqrt[3]{n} \) vertices have degree \( \geq \frac{1}{10} \log n \). \( \square \)
Proof ideas

Observation

If \( p = \frac{(1+\epsilon) \log n}{n} \), then almost all vertices have degree \( \geq \frac{1}{10} \log n \).

First attempt to find rainbow H-cycle:

- Suppose all degrees \( \geq \frac{1}{10} \log n \).
- At each vertex, expose list of colors that appear.
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- Suppose all degrees \( \geq \frac{1}{10} \log n \).
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- Select 3 colors per vertex s.t. all selected colors are different.
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![Diagram](image)

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**Already requires 3n colors.**
Saving the constant factor

**Sprinkling**

Reserve $p' = \frac{\epsilon}{2} \cdot \frac{\log n}{n}$ and $\kappa' = \frac{\epsilon n}{2}$ for 2nd phase.
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Reserve $p' = \frac{\epsilon}{2} \cdot \frac{\log n}{n}$ and $\kappa' = \frac{\epsilon n}{2}$ for 2nd phase.

**Main lemma**

Using only edges and colors from Phase 1, there is a partition into rainbow intervals, such that:

- All intervals have length $L = \frac{14}{\epsilon}$.
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**Main Lemma**

Using only edges and colors from Phase 1, there is a partition into rainbow intervals, such that:

- All intervals have length $L = \frac{14}{\epsilon}$.
- Each $A$-vertex has $\geq \frac{\epsilon^2}{40L} \log n$ $B$-neighbors in Phase 2.
- Each $B$-vertex has $\geq \frac{\epsilon^2}{40L} \log n$ $A$-neighbors in Phase 2.
Final rainbow linking

- Expose Phase 2 colors between $A$- and $B$-vertices.
- Select 2 colors per vertex s.t. all selected colors are different.

Aux. digraph: vertices are intervals; edges oriented $B \rightarrow A$.

Directed H-cycle in $D_{2\text{-in,2\text{-out}}}$ links all intervals via Phase 2.

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- Expose Phase 2 colors between $A$- and $B$-vertices.
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Constructing intervals

Theorem (Ajtai, Komlós, Szemerédi; de la Vega)

Let \( p = \frac{\omega}{n} \), where \( 0 < \omega < \log n - 3 \log \log n \). Then \( G_{n,p} \) has a path of length \( (1 - \frac{1}{\omega})n \) \( \text{whp} \).
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**To obtain intervals:**

- Adapting proof of de la Vega, find rainbow path of length \( n - o(n) \) in Phase 1.
- Break the long path into intervals of length \( L = \frac{14}{\epsilon} \).
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To obtain intervals:

- Adapting proof of de la Vega, find rainbow path of length \( n - o(n) \) in Phase 1.
- Break the long path into intervals of length \( L = \frac{14}{\epsilon} \).
- Absorb all missing vertices into system of intervals, using minimum degree two.

\[ \square \]
**Theorem (Frieze, L.)**

For any fixed $\epsilon > 0$, if $p = \frac{(1+\epsilon) \log n}{n}$, then $G_{n,p}$ contains a rainbow Hamilton cycle *whp* when its edges are randomly colored from $\kappa = (1 + \epsilon) n$ colors.
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Edge-colored random graph process:

- Start with $n$ isolated vertices.
- Each round, add a new edge, selected uniformly at random from all missing edges.
- Randomly color the new edge from a set $C$ of size at least $n$. 
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- Randomly color the new edge from a set $C$ of size at least $n$.

Question

Does a rainbow Hamilton cycle appear as soon as the minimum degree is at least two and at least $n$ colors have arrived?