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For any $t$, there exists $n$ such that every 2-coloring of the edges of $K_n$ has a monochromatic copy of $K_t$. 

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What if there may be arbitrarily many colors? Then is there an $n$ which guarantees a monochromatic copy of $K_t$ or a rainbow copy of $K_t$?

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Every edge can have a different color, so cannot guarantee monochromatic subgraphs.
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Erdős-Rado, 1950:

∀t, ∃n s.t. any edge-coloring of the complete graph on \{1, \ldots, n\}, with \emph{arbitrarily} many colors, has a copy of $K_t$ that is one of:

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- upper lexical: color uniquely determined by larger endpoint
- lower lexical: color uniquely determined by smaller endpoint

Upper lexical coloring:
Constrained Ramsey Number

**Definition:**

Constrained Ramsey number $f(S, T) = \text{minimum } n \text{ such that every edge-coloring of } K_n, \text{ with arbitrarily many colors, has one of:}

- monochromatic copy of $S$
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Existence:

By Canonical Ramsey Theorem, $f(S, T)$ exists iff any sufficiently large lexical coloring also contains either a monochromatic $S$ or a rainbow $T$.

Monochromatic subgraphs of lexical colorings are stars.
Rainbow subgraphs of lexical colorings are forests.

Thus, $f(S, T)$ exists iff $S$ is a star or $T$ is a forest.
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![Diagram]

- Vertices 1 and 3 are connected by a green edge.
- Vertices 3 and 4 are connected by a red edge.
- Vertices 4 and 8 are connected by a blue edge.
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- Only $t - 1$ total colors $\Rightarrow$ no rainbow $T$. 
Upper bounds

Jamison, Jiang, and Ling, 2003: \( f(S, T) \leq O(st^2) \)

- Proof by induction on diameter of \( T \).
- Actually showed \( f(S, T) \leq O(st \cdot \text{diam}(T)) \).
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Proof method: (for contradiction, assume no monochromatic $S$)
- Find an induced subgraph in which most edges can be oriented, such that directed paths are always rainbow.
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The constrained Ramsey number satisfies $f(S, P_t) \leq O(st \log t)$.

That is, for any tree $S$ with $s$ edges and any integer $t$, one can always find either a monochromatic copy of $S$ or a rainbow $t$-path in any edge-coloring of the complete graph on $3600st \log t$ vertices.
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Remarks:
- This is within a logarithmic factor of the previously mentioned lower bound \( f(S, T) \geq \Omega(st) \).
**Main result**

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- Proof significantly extends Wagner’s idea of orienting edges such that directed paths are automatically rainbow.
- We use the concept of median order as an inductive tool, as introduced in Havet and Thomasse (2000).
The proof proceeds by contradiction; suppose that there is no monochromatic $S$ and no rainbow $t$-path.
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**Orientation Lemma:**
There exists a subset $M \subset V$ such that:

- $|M|$ is within a constant factor of the original vertex set size.

Each vertex $v \in M$ is associated with a unique color $c_v$. We may direct most of the edges in $M$ such that if an edge is directed $\overrightarrow{uv}$, then its color was $c_u$. Observation: directed paths are automatically rainbow.
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From colors to directions
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**Observation:** directed paths are automatically rainbow.
**Definition:**

Given a linear ordering $v_1 < v_2 < \cdots < v_n$ of the vertex set of a directed graph, an edge $\overrightarrow{v_i v_j}$ is called *forward* if $i < j$, and *backward* otherwise.

A linear ordering which maximizes the number of forward edges is called a *median order*.

**Remarks:** Always exists, but not necessarily unique. Originally arose in theoretical computer science; NP-hard. Havet and Thomassé (2000) used median orders to give simple proofs of Dean's Conjecture, and Sumner's Conjecture for arborescences.
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**Feedback property**

**Forward bias (helps find directed paths):**

Let $v_1 < \cdots < v_n$ be a median order. Then for every $i < k$,

![Diagram showing forward and backward edges between $v_i$ and $\{v_{i+1}, \ldots, v_k\}$]($v_i$ and $v_k$)

among the edges between $v_i$ and $\{v_{i+1}, \ldots, v_k\}$, there are at least as many forward edges as there are backward edges.

**Proof.** Suppose this is false for some $i < k$, and backward edges outnumber forward edges in the picture. Consider the alternative ordering obtained by moving $v_i$ to the position between $v_k$ and $v_{k+1}$. Edges that switch forward/backward are precisely those between $v_i$ and $\{v_{i+1}, \ldots, v_k\}$. But this increases the total number of forward edges in the graph, contradicting the maximality of the median order. $\square$
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Concluding remarks

Summary of results:

- The constrained Ramsey number $f(S, P_t)$ is upper bounded by $O(st \log t)$.
- That is, for any tree $S$ with $s$ edges and any integer $t$, one can always find either a monochromatic copy of $S$ or a rainbow $t$-path in any edge-coloring of the complete graph on $3600st \log t$ vertices.
- This improves the previous bounds of $O(st^2)$ and $O(s^2t)$. Open problems:
- Remove the logarithmic term that separates our bound from the simple lower bound $f(S, T) \geq \Omega(st)$. We believe that it is an artifact of the proof.
- It would be very interesting to improve the upper bounds for $f(S, T)$ when $T$ is a general tree (instead of a path).
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