Extremal Combinatorics: Homework 3
Due Friday April 29, in class

1. Let $f(k,n)$ be the maximum number of edges in an $n$-vertex graph $G = (V,E)$ whose edge set is the union of $k$ induced matchings. This means $E = E_1 \cup \ldots \cup E_k$, where each $E_i$ is precisely $G[V_i]$ for some subset $V_i \subset V$, with bisections $V_i = V'_i \cup V''_i$ such that there are exactly $|V'_i|$ edges in each $G[V_i]$, and these edges all run between distinct pairs of vertices, one in $V'_i$ and one in $V''_i$. Note in particular that if $E_i = G[V_i]$ is in the above partition, then $G$ has no other edges with both endpoints in $V_i$.

Prove that $f(n,n) = o(n^2)$, i.e., for every $\epsilon > 0$, there exists an $N$ such that for all $n > N$, $f(n,n) \leq \epsilon n^2$.

2. Show that for every $\epsilon > 0$, there exist $c, N > 0$ such that the following holds for every $n > N$. Every $n$-vertex graph with at least $\left(\frac{1}{8} + \epsilon\right)n^2$ edges and independence number $\alpha < cn$ must contain a copy of $K_4$.

3. Let $G$ be an $n$-vertex graph with the property that for every pair of distinct vertices \{x,y\}, the number of vertices $z$ which are adjacent to both $x$ and $y$ is odd. Prove that $n$, the number of vertices of $G$, must also be odd.