11. Integer Polynomials

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1 Problems and well-known statements

1. (Euler.) Prove that there is no polynomial $P(x)$ with integer coefficients and degree at least 1, such that $P(0), P(1), P(2), \ldots$ are all prime.

2. If $P$ is a polynomial with integer coefficients, and $a$ and $b$ are distinct integers, then $P(a) - P(b)$ is divisible by $a - b$.

3. Let $P(x)$ be a polynomial such that $P(n)$ is an integer for every integer $n$. (Note that the coefficients of $P$ are not necessarily integers themselves.) Prove that there are some integers $c_0, \ldots, c_n$ for which

$$P(x) = c_0 \binom{x}{0} + c_1 \binom{x}{1} + \cdots + c_n \binom{x}{n},$$

where $\binom{x}{i}$ is defined for all real $x$ to be $\frac{1}{i!}(x-1)(x-2)\cdots(x-k+1)$.

4. I’m thinking of a polynomial $P$ with nonnegative integer coefficients. As many times as you wish, you’re allowed to give me a real number $a$, and I will evaluate $P(a)$ and tell you the result. Can you figure out what $P$ is (as a polynomial), and if so, how few guesses can you achieve this in?

5. Let $a, b, c$ be three distinct integers, and let $P$ be a polynomial with integer coefficients. Show that the conditions $P(a) = b$, $P(b) = c$, and $P(c) = a$ cannot be satisfied simultaneously.

6. Let $P(x)$ be a polynomial with integer coefficients. Prove that if $P(P(\cdots P(x)\cdots)) = x$ for some integer $x$, where $P$ is repeated $n$ times, then $P(P(x)) = x$.

7. Let $P(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$, where $a, b, c, d, e$ are integers and $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then $r_1r_2$ is also rational.

8. What is the lowest degree monic polynomial (i.e., with leading coefficient equal to 1) for which $P(n) \equiv 0 \pmod{100}$ for every integer $n$?

9. Let $p(x) = x^3 + ax^2 + bx - 1$ and $q(x) = x^3 + cx^2 + dx + 1$ be polynomials with integer coefficients. Suppose that $p(x)$ is irreducible over the rationals, and $a$ is a root of $p(x) = 0$, and $a + 1$ is a root of $q(x) = 0$. Find an expression for another root of $p(x) = 0$ in terms of $a$, but not involving $a, b, c, \text{ or } d$.

10. Let $\alpha$ be a complex $(2^n + 1)$-th root of unity. Prove that there always exist polynomials $p(x)$ and $q(x)$ with integer coefficients, such that

$$p(\alpha)^2 + q(\alpha)^2 = -1.$$ 

11. Let $n$ be a positive odd integer and let $\theta$ be a real number such that $\theta/\pi$ is irrational. Set $a_k = \tan(\theta + k\pi/n), k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1a_2 \cdots a_n}$$

is an integer, and determine its value.
2 Homework

Please write up solutions to two of the statements/problems, to turn in at next week’s meeting. One of them may be a problem that we solved in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.