3. Number theory

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1 Well-known statements

Fermat’s Little Theorem. For every prime \( p \) and any integer \( a \) which is not divisible by \( p \), we have \( a^{p-1} \equiv 1 \pmod{p} \).

Wilson’s Theorem. A positive integer \( n \) is a prime if and only if \((n - 1)! \equiv -1 \pmod{n}\).

Dirichlet’s Theorem. For any two positive integers \( a \) and \( d \) which are relatively prime, the arithmetic progression \( a, a + d, a + 2d, \ldots \) contains infinitely many primes.

Quadratic residues. Let \( p \) be a prime. There are exactly \( \frac{p+1}{2} \) residues \( r \) such that there exist solutions to \( x^2 \equiv r \pmod{p} \).

2 Problems

1. The 9-digit number \( 2^{29} \) has exactly 9 digits, and they are all distinct. Which of the 10 possible digits 0–9 does not appear?

2. There are infinitely many primes of the form \( 4n - 1 \), where \( n \) is an integer.

3. Let \( p \) be a prime, and let \( n \geq k \) be non-negative integers. Prove that

\[
\binom{pn}{pk} \equiv \binom{n}{k} \pmod{p}.
\]

4. Show that for every positive integer \( n \), there is an integer \( N > n \) such that the number \( 5^n \) appears as the last few digits of \( 5^N \). For example, if \( n = 3 \), we have \( 5^3 = 125 \), and \( 5^5 = 3125 \), so \( N = 5 \) would work.

5. Prove that the product of 3 consecutive integers is never a perfect power (i.e., a perfect square, a perfect cube, etc).

6. How many integers \( r \) in \( \{0,1,\ldots,2^n - 1\} \) are there for which there exists an \( x \) where \( x^2 \equiv r \pmod{2^n} \)?

7. Let \( n, a, b \) be positive integers. Prove that \( \gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1 \).

8. A positive integer is written at each integer point in the plane \((\mathbb{Z}^2)\), in such a way that each of these numbers is the arithmetic mean of its four neighbors. Prove that all of the numbers are equal.

9. A triangular number is a positive integer of the form \( n(n+1)/2 \). Prove that \( m \) is the sum of two triangular numbers if and only if \( 4m + 1 \) is the sum of two squares.
3 Homework

Please write up solutions to two of the problems, to turn in at next week’s meeting. One of them may be a problem that we discussed in class. You are encouraged to collaborate with each other. Even if you do not solve a problem, please spend two hours thinking, and submit a list of your ideas.