Course Homepage

http://www.math.cmu.edu/~pikhurko/484/

Instructor: Oleg Pikhurko

Office hours (Wean Hall 7105):
Monday: 15:00–16:00
Thursday: 13:30–15:00

Tentative List of Main Topics

- Matching, Covering and Packing
- Connectivity
- Planar Graphs
- Coloring
- Flows
- Extremal Graph Theory
- Ramsey Theory for Graphs
- Cycles, Paths, and Trees
- Random Graphs

Textbook

This is a very nice and clearly written book containing a lot of material of various levels of
difficulty. A viewable copy is available online at

http://diestel-graph-theory.com

Grade

The TOTAL score consists of

- Weekly homework: 20%

Office Hours (Wean Hall 7105): Mon 15:00–16:00 & Thurs 13:30–15:00
• Unannounced quizzes: 8%
• Three best test scores (out of four), with one test being worth 24%

The final grade will be determined as follows:

\[ \text{A: TOTAL } \geq 85\% \]
\[ \text{B: } 72\% \leq \text{TOTAL } < 85\% \]
\[ \text{C: } 60\% \leq \text{TOTAL } < 72\% \]
\[ \text{D: } 50\% \leq \text{TOTAL } < 60\% \]

Quizes

There will be unannounced quizzes during classes (5–15 quizzes). Their purpose is to test attendance and knowledge. If you are absent during a quiz (for whatever reason), you get score 0 for this quiz. No make-up quizzes will be administered.

Exams

The course will have four in-class exams. They will be given during the regular lecture hour on

• February 11 (Friday)
• March 18 (Friday)
• April 8 (Friday)
• April 29 (Friday)

All exams will be closed-book and closed-notes. No calculators will be permitted. In case you must miss an exam due to illness, family emergency, University-sponsored trip, or religious observance, please notify me as soon as possible. I may require documentation in order to excuse the absence. Failure to do so will result in score 0. Since one worst exam score will be dropped when computing the final grade, no make-up exams will be administered.

Getting Help

If you have any questions, please do one of the following:

1. Talk to me after a class;

Office Hours (Wean Hall 7105): Mon 15:00–16:00 & Thurs 13:30–15:00
2. Come to my office hours;

3. Post your question at the discussion board via the blackboard (I will be regularly checking it and answering all new queries);

4. Finally, if none of the above works, email or phone me.

Blackboard

The course Blackboard is available via http://www.cmu.edu/blackboard/

It gives an access to the gradebook and the discussion forum. You can use Blackboard’s forum to post your questions. Also, other messages related to the course are welcome on the forum. In particular, if you see a question by somebody else and you know the answer or just want to add a comment, you are very welcome to post a follow-up message.

I will be checking the forum regularly and posting replies to any queries that still need answers.
Open Problems

Open Problem 1 Is there an algorithm that decides if the two input graphs $G$ and $H$ are isomorphic with running time bounded by a polynomial function of $v(G)$? (Or in the standard complexity theory language: Is ISOMORPHISM in \textsf{P}?)

Open Problem 2 Find $f(n)$, the smallest $f$ such that any graph of order $n$ and size at least $f$ has a 3-regular subgraph.

Best known bounds (for large $n$): $f(n) \leq Cn \log n$ (Pyber 1985) and $f(n) \geq cn \log \log n$ (Pyber-Rödl-Szemerédi 1995) for some positive constants $c$ and $C$.

Homework Rules

You have to bring written solutions on the due date either to the class or to my mailbox in Wean Hall 6113 by 12:00pm. No extension of the deadline will be given unless you get the prior consent of the grader (whose contact details to be posted later).

You may discuss homework problems with your classmates but you have to write your solutions independently and on your own. Direct copying of someone else’s homework is prohibited. You are not allowed to consult homework solutions from any previous graph theory courses.

The point value of each problem is stated in the brackets. Attempt all questions.

Office Hours on January 24

Since I will be substituting for another instructor at 3:30pm, my office hours for January 24 (Mon) will be 14:00-15:20. Sorry if it causes any inconvenience. -OP

Homework 1. Due January 21


Problem 3 [2] Prove that every sequence $d_1, \ldots, d_n$ of integers such that $d_1 + \ldots + d_n$ is even, $0 \leq d_i \leq n-1$ and $|d_i - d_j| \leq 1$ for every indices $i, j$ is the degree sequence of a graph.
Problem 4 [1+1+1] Which of the following are the degree sequence of some graph?

i) 10, 10, 10, 8, 8, 8, 6, 6, 6, 4, 4, 4,

ii) \( m, m, m, m, m, \underbrace{4, \ldots, 4}_m \), where \( m \geq 5 \),

iii) \( m, m, m, m, \underbrace{4, \ldots, 4}_m \), where \( m \geq 4 \)?

Problem 5 [3] Let \( f(n, d) \) be the largest integer \( f \) such that there is a graph \( G \) of order \( n \) and size \( f \) such that every subgraph \( H \) of \( G \) has minimum degree at most \( d \). Prove that for \( 1 \leq d \leq n \),

\[
f(n, d) = \binom{d}{2} + d(n - d).
\]

[Hint for proving the upper bound: keep removing vertices of degree at most \( d \) until exactly \( d \) vertices are left.]

Grader

Our grader is Dan Crescimanno <dcrescim@andrew.cmu.edu>, phone: 267-218-3532. Please address any questions about homework grades to him.

Homework 2. Due January 28

Problem 6 [2] The union of \( k \) disjoint triangles (plus perhaps an isolated vertex or an isolated edge) shows that the maximum size of a \( P^3 \)-free graph \( G \) is at least \( t(n) \), where we define

\[
t(n) = \begin{cases} 
3k, & n = 3k, \\
3k, & n = 3k + 1, \\
3k + 1, & n = 3k + 2.
\end{cases}
\]

Prove that this is best possible, that is, that every graph with \( n \) vertices and at least \( t(n) + 1 \) edges contains the 3-path as a subgraph.

Problem 7 [3] Show that every graph with non-empty vertex set has a vertex \( x \) and a family of \( \lfloor d(x)/2 \rfloor \) cycles any two of which meet only in the vertex \( x \). Here \( \lfloor \alpha \rfloor \) denotes the largest integer \( k \) such that \( k \leq \alpha \).

Problem 8 [2] Prove that if a graph \( G \) contains two different proper walks \( X = x_0, \ldots, x_k \) and \( Y = y_0, \ldots, y_l \) with \( x_0 = y_0 \) and \( x_k = y_l \), then \( G \) contains a cycle of length at most \( l + k \).

[Hint: use induction on \( k + l \). Argue that we can assume that \( x_1 \neq y_1 \) and \( x_{k-1} \neq y_{l-1} \).]
Problem 9 [1+3+2] The Hoffman-Singleton graph is constructed as follows. We start with 10 vertex-disjoint 5-cycles $P_0, \ldots, P_4, Q_0, \ldots, Q_4$. The vertices of each 5-cycle are labeled by 0, \ldots, 4 as follows.

A vertex $i$ of $P_j$ is adjacent to vertex $i + jk \pmod 5$ of $Q_k$ for all $i, j, k \in \{0, \ldots, 4\}$. (As an example, the extra edges for $i = 2$ and $j = 2$ are shown.)

i) Show that the constructed graph $G$ is 7-regular.

ii) Show that every two vertices $x_1 \neq x_2$ of $G$ are either adjacent or connected by a path of length 2. [Hint: to reduce the number of cases, note that for any $j, k$ the subgraph induced by the union of $P_j$ and $Q_k$ is isomorphic to the Petersen graph.]

iii) Assuming the results of Items i) and ii) (or any other way), show that $G$ has no cycle of length less than 5.

Problem 10 [1+2] Let the diamond graph $D$ be obtained from $K^4$ by removing one edge. (Thus $|D| = 4$ and $\|D\| = 5$.)

i) Find the total number of all possible walks of length 4 in $D$. (We allow non-proper walks in Items i) and ii).)

ii) Find the total number of all possible closed walks of length 8 in $D$, that is, walks in which the initial and final points coincide.

Open Problems

Open Problem 3 Given integers $k \geq 2$ and $n \geq k$, let $c(n,k)$ denote the maximum possible number of edges in an $n$-vertex graph which has no $(k+1)$-connected subgraph. Mader
conjectured in 1972 that for every $k \geq 1$, if $n$ is sufficiently large then
\[ c(n, k) \leq \frac{3k - 1}{2}(n - k). \]

Yuster (2003) proved that when $n \geq n_0(k)$ is sufficiently large then
\[ c(n, k) \leq \frac{193k}{120}(n - k) < 1.61k(n - k), \]
thereby coming rather close to the conjectured bound.

Exam 1

Exam 1 will take place during the regular class hour on February 11. It is closed book and notes. All material that was covered in class until the lecture on February 4 (inclusive) is examinable. You should be able to state the results and reproduce the proofs that we did in class, and know how to apply this knowledge to problem solving.

Office Hours Next Week

There will be no office hours on February 10 (Thurs). Instead, I will have office hours 2-4pm on February 9 (Wen). Sorry if this causes any inconvenience.

If this helps with preparation for Exam 1, you may bring your Homework 4 to my office hours on February 9 and collect my solutions, provided you promise not to show or discuss them with anybody else before Exam 1. Otherwise, please bring your solutions to Exam 1.

Homework 3. Due February 4

Problem 11 [1] Let $n \geq 0$ be an integer such that $\sqrt{n}$ is rational. Prove that $\sqrt{n}$ must be an integer.

Problem 12 [1+2+2+2] Suppose that $G$ is a $k$-regular graph of order $n \geq 2$ such that every two vertices have exactly one common neighbor. Prove that $k = 2$ and $n = 3$ (that is, that $G = K^3$) as follows.

i) Show that $A^2 = J + (k - 1)I$, where $A$ is the adjacency matrix of $G$, $J$ is the all-1 matrix, and $I$ is the identity matrix (each of dimension $n \times n$).

ii) Show that $n = k^2 - k + 1$.

iii) Show that $A$ has eigenvalue $k$ with multiplicity 1, with the remaining $n - 1$ eigenvalues being $\pm \sqrt{k - 1}$.
iv) Derive the required statement by looking at the trace of $A$.

**Problem 13** [2] Determine the connectivity $\kappa(Q^n)$ of the $n$-dimensional cube (whose vertex set consists of all binary $n$-sequences and we connect two sequences if they differ in exactly one place).

[Hint: When you prove the lower bound, use induction on $n$ and the fact that $Q^n$ consists of two copies of $Q^{n-1}$ connected by a perfect matching.]

**Problem 14** [2] Suppose that $k \geq 2$ is fixed and $k$ divides $n$. Let $G$ be the graph of order $n$ defined as follows. Partition its vertex set $V(G) = U_1 \cup \ldots \cup U_m$ into disjoint $k$-sets, where $m = n/k$. Put a complete graph on each $U_i$ with $2 \leq i \leq m$ and put all edges between $U_1$ and $V \setminus U_1$.

Show that $G$ has no $(k + 1)$-connected subgraph $H$.

**Problem 15** [2] Let $n \geq 2$. Prove that a sequence $(d_1, \ldots, d_n)$ of positive integers is a degree sequence of a tree if and only if $\sum_{i=1}^{n} d_i = 2(n - 1)$.

**Homework 4. Due February 11**

**Problem 16** [1+2] i) Draw all 6 non-isomorphic unlabeled trees on 6 vertices.

ii) How many non-isomorphic unlabeled forests are there on 6 vertices? (You do not need to draw them all.)

**Problem 17** [1+1+1] Recall that the Prüfer-Wilson code of a tree $T$ on $[n] = \{1, \ldots, n\}$ is obtained as follows. Let $n$ be the root and define the corresponding parent-child relation. Repeat while there are at least 3 vertices left:

- remove the smallest leaf $c$,
- write down the parent $p$ of $c$.

i) Describe the converse, that is, how to recover the tree from its code. You are not required to prove anything here. Also, note that the root is $n$ (not 1 as we had in class).

ii) Show that the leaves of $T$ are exactly those elements $[n]$ that do not appear in $T$’s code $\{p_1, \ldots, p_{n-2}\}$. (If $n$ has degree 1, it is also counted as a leaf.)

iii) How many trees on $[n]$ have exactly 3 leaves?

**Problem 18** [1+1+2+2] A fractional matching in a graph $G = (V, E)$ is an assignment $w : E(G) \to \mathbb{R}$ of non-negative reals (called weights) to edges so that for every vertex $x$
we have \( \sum_{y \in N(x)} w(xy) \leq 1 \). The fractional matching number \( \mu'(G) \) is the maximum of \( \sum_{xy \in E} w(xy) \) over all fractional matchings \( w \).

i) Show that \( \mu(G) \leq \mu'(G) \) for every graph \( G \).

ii) Determine \( \mu'(C^k) \) for each integer \( k \geq 3 \), where \( C^k \) is the \( k \)-cycle.

iii) Prove that \( \mu'(G) = \mu(G) \) for every bipartite graph \( G \).

iv) Prove that \( \mu'(G) \leq 2\mu(G) \) for every graph \( G \).

**Remark:** Note that \( \mu'(G) \) is the value of some linear program; thus we can determine it in polynomial time. By Item iii), this gives yet another algorithm that computes \( \mu(G) \) for bipartite \( G \).

**Problem 19** [1+3] Let \( G = (V, E) \) be a graph and \( M \subseteq E \) be a matching. A path \( P = x_0, \ldots, x_k \) is called augmenting if

- \( x_{i-1}x_i \in E \setminus M \) for every odd \( i \in [k] \);
- \( x_{i-1}x_i \in M \) for every even \( i \in [k] \);
- Neither \( x_0 \) nor \( x_k \) is incident to an edge of \( M \).

i) Show that if an augmenting path exists, then \( M \) is not maximum (that is, \( G \) has another matching \( M' \) with \( |M'| > |M| \)).

ii) Show that if no augmenting path exists, then \( |M| = \mu(G) \). [Hint: what do we get when we take the XOR (exclusive OR) of two matchings of different sizes?]

**Practice Exercises on the Hungarian Algorithm**

Let \( G \) be the complete bipartite graph with parts \( A = \{a_1, \ldots, a_m\} \) and \( B = \{b_1, \ldots, b_n\} \), where the weight of an edge \( \{a_i, b_j\} \) is \( c_{ij} \). Solve the following three problems using the Hungarian Algorithm (and include all intermediate steps).

**Exercise 1** Let \( m = n = 5 \). Find the minimum weight of a perfect matching in \( G \) if

\[
(c_{ij}) = \begin{bmatrix}
2 & 4 & 3 & 2 & 1 \\
3 & 4 & 4 & 5 & 2 \\
1 & 4 & 1 & 4 & 5 \\
3 & 8 & 5 & 3 & 8 \\
4 & 6 & 6 & 2 & 3
\end{bmatrix}.
\]
Exercise 2 Let \( m = n = 4 \). Find the minimum weight of a perfect matching in \( G \) if
\[
(c_{ij}) = \begin{bmatrix}
7 & 2 & 5 & 4 \\
6 & 5 & 4 & 3 \\
4 & 1 & 5 & 2 \\
2 & 1 & 1 & 2
\end{bmatrix}.
\]

Exercise 3 Let \( m = 4 \) and \( n = 5 \). Find the maximum weight of an \( A \)-saturating matching if
\[
(c_{ij}) = \begin{bmatrix}
9 & 8 & 7 & 6 & 5 \\
9 & 7 & 5 & 3 & 1 \\
9 & 6 & 3 & 0 & 3 \\
9 & 5 & 1 & 5 & 9
\end{bmatrix}.
\]

Open Problems

A \( k \)-hypergraph \( G \) is a collection of sets (called edges), each consisting of \( k \) vertices. The vertex cover number \( \tau(G) \) is defined as the smallest size of a set \( X \) of vertices such that every edge intersects \( X \) in at least one vertex. The matching number \( \mu(G) \) is the maximum number of disjoint edges. Also, \( G \) is called \( k \)-partite if all vertices can be partitioned into \( k \) parts \( V_1, \ldots, V_k \) such that every edge intersects each part in exactly one vertex.

Open Problem 4 (Ryser’s Conjecture (1970s)) If \( G \) is a \( k \)-partite \( k \)-hypergraph, then
\[
\tau(G) \leq (k - 1)\mu(G).
\]

This conjecture is true for \( k = 2 \) (König’s theorem) and it was proved for \( k = 3 \) by Aharoni in 2001. Note the easy bound \( \tau(G) \leq k\mu(G) \) valid for an arbitrary \( k \)-hypergraph \( G \).

Solution to Quiz 2

Question. Characterize graphs such that every edge is a bridge.

Answer. Any such \( G \) has to be a forest (that is, acyclic). Indeed, if we have a cycle \( C \) then the removal of any edge \( e \) of \( C \) does not increase the number of components. (If some path used \( e \), we can just re-route it using other edges of \( C \).)

Conversely, the removal of any edge \( xy \) in a forest disconnects \( x \) from \( y \) and increases the number of components.

Thus the answer is: all forests.
Homework 5. Due February 18

**Problem 20** [2+2] Let \( r \geq 2 \) be integer.

i) Show that for any \( r \) integers \( n_1, \ldots, n_r \) there are \( i \) and \( j \) such that \( 1 \leq i \leq j \leq r \) and \( n_i + n_{i+1} + \ldots + n_j \) is divisible by \( r \).

ii) Show that if the minimum degree of a graph \( G \) is at least \( r + 1 \), then \( G \) contains a cycle of length equal to 2 modulo \( r \). [Hint: Recycle the main idea of HW Problem 7.]

**Problem 21** [1+1] Determine the number of perfect matchings in

i) \( K^{2n} \);

ii) \( K_{n,n} \), where \( K_{n_1,n_2} \) denotes the complete bipartite graph with parts of sizes \( n_1 \) and \( n_1 \). (Thus \( |K_{n_1,n_2}| = n_1 + n_2 \) and \( \|K_{n_1,n_2}\| = n_1n_2 \).)

**Problem 22** [2] Let \( G = (V, E) \) be a graph. A set \( A \subseteq V \) is independent if no edge lies inside \( A \) (or equivalently, if the subgraph \( G[A] \) induced by \( A \) has no edges). Let \( A, B \subseteq V \) be two independent sets of the largest size. Prove that the bipartite subgraph of \( G \) with parts \( A \setminus B \) and \( B \setminus A \) has a perfect matching.

**Problem 23** [2] Let \( k \geq 1 \) be integer. Let \( G \) be a bipartite graph with parts \( A \cup B \). Show that there is a subgraph \( H \subseteq G \) with \( d_H(x) = 1 \) for each \( x \in A \) and \( d_H(x) \leq k \) for each \( x \in B \) if and only if \( |N(X)| \geq |X|/k \) for each \( X \subseteq A \). (Here \( d_H(x) \) denotes the degree of a vertex \( x \) with respect to graph \( H \).)

**Problem 24** [1+1] Let \( G \) be the complete bipartite graph with parts \( \{x_1, \ldots, x_4\} \) and \( \{y_1, \ldots, y_4\} \) where the edge costs are encoded by the following matrix:

\[
\begin{bmatrix}
8 & 8 & 0 & 9 \\
1 & 0 & 0 & 9 \\
9 & 4 & 0 & 6 \\
3 & 9 & 0 & 0 \\
\end{bmatrix}
\]

Using the Hungarian algorithm, find a perfect matching of

i) the minimum cost;

ii) the maximum cost.

**Exam 2**

Exam 2 will take place during the regular class hour on March 18. It is closed book and notes. All material that was covered in class until the lecture on March 2 (inclusive) is examinable. You should be able to state the results and reproduce the proofs that we did in class, and know how to apply this knowledge to problem solving.
Office Hours Week March 14–18

There will be no office hours on March 17 (Thurs). Instead, I will have office hours 9–11am on March 16 (Wen). Sorry if this causes any inconvenience.

Homework 6. Due February 25

Problem 25 [1+1] Let $G$ be the complete bipartite graph with parts $A = \{a_1, \ldots, a_6\}$ and $B = \{b_1, \ldots, b_6\}$, where the edge weights are encoded by the following matrix:

$$
\begin{pmatrix}
1 & 2 & 3 & 0 & 9 & 9 \\
2 & 3 & 4 & 5 & 9 & 0 \\
1 & 0 & 2 & 3 & 9 & 8 \\
0 & 8 & 8 & 8 & 0 & 9 \\
6 & 7 & 0 & 2 & 9 & 2 \\
9 & 9 & 9 & 9 & 9 & 0 \\
\end{pmatrix}
$$

Find i) the maximum; ii) the minimum weight of a perfect matching of $G$, using the Hungarian Algorithm.

Problem 26 [1+1+3+1] Suppose we want to bijectively assign $n$ workers to $n$ jobs with $A = (a_{ij})_{i,j=1}^n$ being the $n \times n$ cost matrix. Then the smallest cost $M$ is the minimum of $L(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_{ij}$ given restrictions

$$\forall i \in [n] \quad R_i \text{ holds: } \sum_{j=1}^n x_{ij} = 1$$

$$\forall j \in [n] \quad C_j \text{ holds: } \sum_{i=1}^n x_{ij} = 1$$

over non-negative integers $x_{ij}$, $i,j \in [n]$. The LP relaxation is to determine $M'$, the minimum of $L(x)$ given Conditions $R_i$ and $C_j$ except $x_{ij}$ can now be an arbitrary non-negative real. Clearly, $M' \leq M$. Prove that $M' = M$ as follows.

i) Show that if we have reals $r_i, c_i$ for $i \in [n]$ such that $r_i + c_j \leq a_{ij}$ for every $i,j \in [n]$, then $M' \geq \sum_{i=1}^n (r_i + c_i)$.

Assume (without proof) the following LP duality property: there are feasible reals $x_{ij} \geq 0$ satisfying all Conditions $R_i$ and $C_j$ and there are reals $r_i, c_i$ satisfying Item i) such that $\sum_{i=1}^n (r_i + c_i) = L(x)$. Fix one such choice of $x_{ij}, r_i, c_i$ for the remainder of the problem.

ii) Deduce that $L(x) = M'$. Moreover, show that if $x_{ij} > 0$ for some $i,j \in [n]$ then $r_i + c_j = a_{ij}$.

iii) Using Hall’s or König’s Theorem prove that there is a permutation $\sigma : [n] \to [n]$ such that $x_{i,\sigma(i)} > 0$ for every $i \in [n]$.
Conclude from Items i–iii) that $M = M'$.

**Problem 27** [1] Let $G$ be a bipartite graph with parts $A$ and $B$ of equal size: $|A| = |B|$. Give a direct proof that Tutte's 1-factor condition implies Hall's condition.

**Problem 28** [1] Find a 3-regular graph without a perfect matching.

**Homework 7. Due March 16 (Wednesday)**

**Problem 29** [2] Is the following statement true? Justify your answer.

Let $G$ be a bipartite graph with parts $A$ and $B$ which may now have infinitely many vertices (and we allow infinite degrees). Then $G$ has a matching saturating all vertices of $A$ if and only if for every $X \subseteq A$ we have $|N(X)| \geq |X|$. (For possibly infinite sets $X$ and $Y$, the inequality $|Y| \geq |X|$ means that there is an injective map from $X$ into $Y$).

**Problem 30** [3] (Defect Form of the Tutte 1-Factor Theorem) Let $G = (V,E)$ be a graph and $k$ be an integer such that $|G| - k$ is even. Deduce from the Tutte 1-Factor Theorem that a graph $G$ has a matching covering all but at most $k$ vertices if and only if $q(G - S) \leq |S| + k$ for every set $S \subseteq V$. Here $q(H)$ denotes the number of components of $H$ of odd order.

**Problem 31** [3+2] Let $n \geq 5$. What is

i) the maximum number of edge-disjoint spanning trees

ii) the minimum number of forests covering all edges

for the graph $K_{3,n}$, the complete bipartite graph with parts of sizes 3 and $n$?

**Problem 32** [3+2+2+2+2] Let $K^n$ be the complete graph on $n$ vertices. Let $n \geq 2$. Prove each of the following claims.

i) The maximum number of edge-disjoint spanning trees in $K^n$ is $\lfloor n/2 \rfloor$.

ii) The minimum number of forests that cover all edges of $K^n$ is $\lceil n/2 \rceil$.

Furthermore, determine

iii) the maximum number of edge-disjoint spanning trees;

iv) the minimum number of forests that cover all edges;

v) the maximum size of a matching
in the following graph $G$:

Thus $G$ has $5n + 2$ vertices and $5 \binom{n}{2} + 10$ edges (and the neighborhoods of $x$ and $y$ are disjoint). You have to justify your answer. You are allowed to use the results of Items i) and ii) here.

**Problem 33** [2] Given $n \geq 5$, what is the smallest number of edges in a graph of order $n$ whose edges cannot be covered by 2 forests?

**Exam 3**

Exam 3 will take place during the regular class hour on April 8. It is closed book and notes. All material that was covered in class until the lecture on April 2 (inclusive) is examinable. You should be able to state the results and reproduce the proofs that we did in class, and know how to apply this knowledge to problem solving.

If this helps with preparation for the exam, you may bring your Homework 9 to the office hours on April 7 and collect my solutions, provided you promise not to show or discuss them with anybody else before the exam.

**Homework 8. Due April 1**

**Problem 34** [1] Using the Matrix Tree Theorem, find the number of spanning trees in the graph $G$ obtained from the complete graph $K^8$ by removing a perfect matching. Thus $\|G\| = \binom{8}{2} - 4 = 24$. You may use any software for manipulating matrices.

**Problem 35** [2] Deduce from the Gallai-Milgram Theorem (Theorem 2.5.1 in Diestel) that every bipartite graph $G$ has a matching $M$ and a vertex cover $C \subseteq V(G)$ such that $|M| = |C|$. In other words, you have to deduce the “harder” part of the König Theorem (Theorem 2.1.1 in Diestel).

**Problem 36** [1+1] Let $c(H)$ be the number of components of a graph $H$. (Thus $c(H) = 1$ if and only if $H$ is connected.)
i) Show that if $G = (V, E)$ is $k$-edge-connected, where $k > 0$, and if $F$ is a set of $k$ edges, then $c((V, E \setminus F)) \leq 2$.

ii) For $k > 0$, find a $k$-connected graph $G$ and a set $X$ of $k$ vertices of $G$ such that $c(G - X) > 2$.

**Problem 37** [3] By adopting the proof of Lemma 3.2.4 in Diestel or otherwise, prove that every 2-connected graph $G$ with at least 4 vertices has an edge $e$ such that $G/e$ is 2-connected.

**Problem 38** [3] Given integers $n$ and $k$ such that $k \leq n - 1$ and $n + k$ is odd, what is the smallest $m$ such that every graph of order $n$ and minimum degree $m$ is $k$-connected?

**Problem 39** [2+1]

i) Show that if $G$ has $n$ vertices and minimum degree $\delta(G) \geq n - 2$, then $\kappa(G) = \delta(G)$.

ii) For each $n \geq 4$, find a graph $G$ such that $|G| = n$, $\delta(G) = n - 3$, and $\kappa(G) < \delta(G)$.

**Extra Credit**

If you wish to earn up to 5 homework points as extra credit, please submit (along with Homework 8) a joke, a cartoon, or a short funny story about any mathematical concepts that we have covered in the class (or about hedgehogs).

But note that if the Instructor appears in your submission, then your extra credit can be negative!
Homework 9. Due April 8

Problem 40 [2] Show that every 3-regular 3-edge-connected graph is 3-connected.

Problem 41 [3] Let $G$ be a graph that is not complete. Prove that $G$ is $k$-connected if and only if there are $k$ independent paths between any two non-adjacent vertices. (You can use any theorems from Diestel’s book.)

Problem 42 [2] Let $G = (V, E)$ be a $k$-connected graph and let $S \subseteq V$ be a set of at least $k$ vertices. Let $H$ be obtained from $G$ by adding a new vertex $w$ and joining $w$ to the vertices of $S$. Prove that $H$ is $k$-connected.

Problem 43 [1+2] A collection of circles and lines in the plane is in general position if every two lines intersect in exactly one point, every two circles (or a circle and a line) intersect in exactly two points, while no point belongs to more than 2 figures. Determine the number of regions (including infinite regions) for the following collections.

(i) $c$ circles in general position, where $c \geq 1$;

(ii) $l$ lines and $c$ circles in general position, where $c, l \geq 1$.

Problem 44 [2] Argue that a planar bipartite graph on $n \geq 3$ nodes has at most $2n - 4$ edges.

Quiz 3

Question. Find a subdivision of $K_{3,3}$ inside the Petersen graph.

Answer.

Office Hours (Wean Hall 7105): Mon 15:00–16:00 & Thurs 13:30–15:00
Exam 4

Exam 4 will take place during the regular class hour on April 29 (Fri). It is closed book and notes. All material that was covered in class until the lecture on April 22 (inclusive) is examinable. You should be able to state the results and reproduce the proofs that we did in class, and know how to apply this knowledge to problem solving.

If this helps with preparation for the exam, you may bring your homework to the office hours on April 28 and collect my solutions, provided you promise not to show or discuss them with anybody else before the exam.

Homework Queries

If you have any questions about your HW grades, please see our grader Dan Crescimanno (cellphone: 267-218-3532). In particular, he will be available 3–4pm on April 24 (Mon) at the chairs near the elevator, 7th floor of Wean Hall.

Quiz 4

Question. Let $G$ be obtained from the triangle $K^3$ by adding a pendant edge. (Thus $|G| = \|G\| = 4$.) Find the chromatic polynomial of $G$.

Answer. The easiest way is to color vertices of $G$ one by one. If we start with the leaf, then color the vertex of degree 3, and then the two remaining vertices (each of degree 2), the number of choices is respectively $k$, $k-1$, $k-1$, and $k-2$. Hence,

$$P_G(k) = k(k-1)^2(k-2) = k^4 - 4k^3 + 5k^2 - 2k.$$ 

Alternatively, if $p_l$ denotes the number of partitions of $V(G)$ into $l$ non-empty independent sets, then $p_1 = p_2 = 0$, $p_3 = 2$, and $p_4 = 1$. Thus $P_G(k) = 2k(k-1)(k-2) + k(k-1)(k-2)(k-3)$, giving the same answer after simplifications.

Homework 10. Due April 20

Problem 45 [1+3+2] i) Let $G = (V,E)$ be a $k$-connected graph. Let $A \subseteq V$ and $a \in V$ be arbitrary with $a \notin A$. Prove that $G$ contains at least $\min(k, |A|) a - A$ paths that are disjoint except for the common vertex $a$.

ii) Let $k \geq 2$. Show that in a $k$-connected graph any set $S$ of $k$ vertices lies on a common cycle. Hint: Take a cycle $C$ containing the maximum number of elements of $S$. If some $a \in S$ does not belong to $V(C)$, apply Item i) to $a$ and $A = V(C)$ and find a “better” cycle.
iii) For every $k \geq 2$ give an example of a $k$-connected graph and a set of $k + 1$ vertices that do not lie on a common cycle.

**Problem 46 [2+1]**

i) Using Nash-William’s Theorem prove that the edges of every planar graph can be partitioned into at most 3 forests.

ii) Give an example that shows that 2 forests are not enough in general.

**Problem 47 [1]** Does every planar graph have a drawing with all inner faces convex?

**Problem 48 [1+1]**

i) Find a planar graph all of whose vertices have degree 5.

ii) Prove that any such graph has at least 12 vertices.

**Problem 49 [2]**

Show that a plane graph with $v \geq 3$ vertices has at most $2v - 4$ faces.

**Problem 50 [1+1+2]**

A graph $G$ is critically $k$-chromatic if $\chi(G) = k$ but for every vertex $x$ we have $\chi(G - x) < k$.

i) Show that every $k$-chromatic graph $G$ has a critically $k$-chromatic induced subgraph.

ii) Show that every critically $k$-chromatic graph $H$ has minimum degree at least $k - 1$. (Thus, Items i) and ii) give another proof of the result that every $k$-chromatic graph $G$ has a subgraph of minimum degree at least $k - 1$.)

iii) Determine all critical 3-chromatic graphs.

**Homework 11. Due April 29**

Recall that $P_G(k)$ is the chromatic polynomial of $G$, that is, if $k > 0$ is an integer, then $P_G(k)$ is the number of way to color the vertices of $G$ with $k$ colors.

**Problem 51 [1]**

Let $n \geq 3$. Show that for the cycle $C^n$ of length $n$, we have $P_{C^n}(x) = (x - 1)^n + (-1)^n(x - 1)$.

**Problem 52 [1+1+2]**

Let $G$ have $n \geq 1$ vertices and let $P_G(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be its chromatic polynomial.

i) Prove that $a_0 = 0$.

ii) What is $a_{n-1}$?

iii) Prove that $a_{n-2} = \binom{|G|}{2} - t(G)$, where $t(G)$ is the number of triangles in $G$.

**Problem 53 [1]** Prove that, in any $k$-coloring of a $k$-chromatic graph and every color $c$ there is a vertex $x$ of color $c$ which is adjacent to vertices of every other color.
Problem 54 \[2\] Let $G = K_{n,n}$ be the complete bipartite graph with $n = \binom{2k-1}{k}$. Show how to assign a list $L_x$ of some $k$ colors to each vertex $x \in V(G)$ so that there is no proper coloring $c$ of $G$ with $c(x) \in L_x$. (Thus, in the terminology of Section 5.4 in Diestel, this states that the list-chromatic number of $G$ is larger than $k$.)

Problem 55 \[3\] Prove that for every graph $F$ and integers $n \leq m$, we have

$$
\frac{\text{ex}(n, F)}{\binom{n}{2}} \geq \frac{\text{ex}(m, F)}{\binom{m}{2}}
$$

Quiz 5

Question. What is $\text{ex}(11, K^4)$, the maximum size of a $K^4$-free graph on 11 vertices?

Answer. By Turán’s theorem, the answer is the size of $T^3(11)$, the complete 3-partite graph on 11 vertices with almost equal part sizes. Here, the part sizes are 3, 4, and 4, so the answer is

$$
\|T^3(11)\| = 3 \cdot 4 + 3 \cdot 4 + 4 \cdot 4 = 40.
$$