A-Quasiconvexity: Relaxation and Homogenization

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Abstract

Integral representation of relaxed energies and of Γ-limits of functionals

\[(u, v) \mapsto \int_{\Omega} f(x, u(x), v(x)) \, dx\]

are obtained when sequences of fields \(v\) may develop oscillations and are constrained to satisfy a system of first order linear partial differential equations. This framework includes the treatment of divergence-free fields, Maxwell’s equations in micromagnetics, and curl-free fields. In the latter case classical relaxation theorems in \(W^{1,p}\) are recovered.

Key words. A-quasiconvexity, equi-integrability, Young measure, relaxation, Γ-convergence, homogenization

AMS subject classifications. 35D99, 35E99, 49J45

1 Introduction

In a recent paper Fonseca and Müller [22] have proved that \(\mathcal{A}\)-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

\[(u, v) \mapsto \int_{\Omega} f(x, u(x), v(x)) \, dx,\]

whenever \(f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \rightarrow [0, \infty)\) is a Carathéodory integrand satisfying

\[0 \leq f(x, u, v) \leq a(x, u) (1 + |v|^p),\]

for a.e. \(x \in \Omega\) and all \((u, v) \in \mathbb{R}^m \times \mathbb{R}^d\), where \(1 \leq q < \infty\), \(a \in L^\infty(\Omega \times \mathbb{R}; [0, \infty))\), \(\Omega \subset \mathbb{R}^N\) is open, bounded, \(u_n \rightharpoonup u\) in measure, \(v_n \rightharpoonup v\) in \(L^q(\Omega; \mathbb{R}^d)\) and \(Av_n \rightharpoonup 0\) in \(W^{-1,q}(\Omega; \mathbb{R}^d)\) (see also [14]). Here, and in what follows, following [32]

\[\mathcal{A} : L^q(\Omega; \mathbb{R}^d) \rightarrow W^{-1,q}(\Omega; \mathbb{R}^d), \quad Av := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i},\]

is a constant–rank, first order linear partial differential operator, with \(A^{(i)} : \mathbb{R}^d \rightarrow \mathbb{R}^d\) linear transformations, \(i = 1, \ldots, N\). We recall that \(\mathcal{A}\) satisfies the constant-rank property if there exists \(r \in \mathbb{N}\) such that

\[\text{rank } Av = r \quad \text{for all } w \in S^{N-1}, \quad (1.1)\]

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where
\[ Aw := \sum_{i=1}^{N} w_i A^{(i)}, \quad w \in \mathbb{R}^N. \]

A function \( f : \mathbb{R}^d \to \mathbb{R} \) is said to be \( A \)-quasiconvex if
\[ f(v) \leq \int_Q f(v + w(y)) \, dy \]
for all \( v \in \mathbb{R}^d \) and all \( w \in C_{1,\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \) such that \( Aw = 0 \) and \( \int_Q w(y) \, dy = 0 \). Here \( Q \) denotes the unit cube in \( \mathbb{R}^N \), and the space \( C_{1,\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \) is introduced in Section 2.

The relevance of this general framework, as emphasized by Tartar (see [32, 34, 35, 36, 37, 38, 39]), lies on the fact that in continuum mechanics and electromagnetism PDEs other than \( \text{curl} \, v = 0 \) arise naturally, and this calls for a relaxation theory which encompasses PDE constraints of the type \( Au = 0 \). Some important examples included in this general setting are given by:

(a) [Unconstrained Fields]
\[ Au = 0. \]

Here, due to Jensen’s inequality \( A \)-quasiconvexity reduces to convexity.
(b) [Divergence Free Fields]
\[ Au = 0 \quad \text{if and only if} \quad \text{div} \, v = 0, \]
where \( v : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N \) (see [33]).

(c) [Magnetostatics Equations]
\[ A \left( \frac{m}{h} \right) := \left( \text{div}(m + h), \text{curl} \, h \right) = 0, \]
where \( m : \mathbb{R}^3 \to \mathbb{R}^3 \) is the magnetization and \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) is the induced magnetic field (see [17, 38]); often these are also called Maxwell’s Equations in the micromagnetics literature.

(d) [Gradients]
\[ Au = 0 \quad \text{if and only if} \quad \text{curl} \, v = 0. \]

Note that \( w \in C_{1,\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \) is such that \( \text{curl} \, w = 0 \) and \( \int_Q w(y) \, dy = 0 \) if and only if there exists \( \varphi \in C_{1,\text{per}}(\mathbb{R}^N; \mathbb{R}^n) \) such that \( \nabla \varphi = v \), where \( d = n \times N \). Thus in this case we recover the well-known notion of quasiconvexity introduced by Morrey [30],

(e) [Higher Order Gradients]
Replacing the target space \( \mathbb{R}^d \) by an appropriate finite dimensional vector space \( E^n \), it is possible to find a first order linear partial differential operator \( A \) such that \( v \in L^p(\Omega; E^n) \) and \( Au = 0 \) if and only if there exists \( \varphi \in W^{1,q}(\Omega; \mathbb{R}^n) \) such that \( v = \nabla \varphi \) (see Theorem 1.3).

This paper is divided into two parts. In the first part we give an integral representation formula for the relaxed energy in the context of \( A \)-quasiconvexity. Precisely, let \( 1 \leq p < \infty \) and \( 1 < q < \infty \), and consider the functional
\[ F : L^p(\Omega; \mathbb{R}^m) \times L^q(\Omega; \mathbb{R}^d) \times O(\Omega) \to [0, \infty) \]
defined by
\[ F((u, v); D) := \int_D f(x, u(x), v(x)) \, dx, \]
where \( O(\Omega) \) is the collection of all open subsets of \( \Omega \), and the density \( f \) satisfies the following hypothesis:

\[ (H) \ f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, \infty) \text{ is Carathéodory function satisfying} \]
\[ 0 \leq f(x, u, v) \leq C (1 + |u|^p + |v|^q) \]

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for a.e. $x \in \Omega$ and all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^d$, and where $C > 0$.

For $D \in \mathcal{O}(\Omega)$ and $(u, v) \in L^p(\Omega; \mathbb{R}^m) \times (L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A})$ define

$$
\mathcal{F}((u, v); D) := \inf \left\{ \liminf_{n \to \infty} F((u_n, v_n); D) : (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d), \right.
$$
$$
\left. u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \quad v_n \to v \text{ in } L^q(D; \mathbb{R}^d), \quad \mathcal{A}v_n \to 0 \text{ in } W^{-1,q}(D; \mathbb{R}^d) \right\}.
$$

(1.2)

It turns out that the condition $\mathcal{A}v_n \to 0$ imposed in (1.2) may be replaced by requiring that $v_n$ do satisfy the homogeneous PDE $\mathcal{A}v = 0$. Precisely, and in view of Lemma 3.1 and Corollary 3.2 below, it can be shown that

$$
\mathcal{F}((u, v); D) = \inf \left\{ \liminf_{n \to \infty} F((u, v_n); D) : v_n \in L^q(D; \mathbb{R}^d), v_n \to v \text{ in } L^q(D; \mathbb{R}^d), \mathcal{A}v_n = 0 \right\},
$$

and thus

$$
\mathcal{F}((u, v); D) = \inf \left\{ \liminf_{n \to \infty} F((u_n, v_n); D) : (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d), \right.
$$
$$
\left. u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \quad v_n \to v \text{ in } L^q(D; \mathbb{R}^d), \quad \mathcal{A}v_n = 0 \right\} =: \mathcal{F}_0((u, v); D).
$$

(1.3)

The first main result of the paper is given by the following theorem:

**Theorem 1.1.** Under condition (H) and the constant-rank hypothesis (1.1), for all $D \in \mathcal{O}(\Omega)$, $u \in L^p(\Omega; \mathbb{R}^m)$, and $v \in L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}$, we have

$$
\mathcal{F}((u, v); D) = \int_D \mathcal{Q}_A f(x, u(x), v(x)) \, dx
$$

where, for each fixed $(x, u) \in \Omega \times \mathbb{R}^m$, the function $\mathcal{Q}_A f(x, u, \cdot)$ is the $\mathcal{A}$-quasiconvexification of $f(x, u, \cdot)$, namely

$$
\mathcal{Q}_A f(x, u, v) := \inf \left\{ \int_Q f(x, u + w(y)) \, dy : w \in C^\infty_{1,\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A}, \int_Q w(y) \, dy = 0 \right\}
$$

for all $v \in \mathbb{R}^d$.

**Remarks 1.2.** (i) Note that in the degenerate case where $\mathcal{A} = 0$, $\mathcal{A}$-quasiconvex functions are convex and Theorem 1.1 together with condition (1.4) yield a convex relaxation result with respect to $L^p \times L^q$ (weak) convergence. See the monograph of Buttazzo [12] for related results in this context.

(ii) If the function $f$ also satisfies a growth condition of order $q$ from below in the variable $v$, that is

$$
f(x, u, v) \geq \frac{1}{C} |v|^q - C
$$

(1.4)

for a.e. $x \in \Omega$ and all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^d$, then a simple diagonalization argument shows that $(u, v) \mapsto \mathcal{F}((u, v); D)$ is $L^p \times (L^q$-weak) lower semicontinuous, i.e.,

$$
\int_D \mathcal{Q}_A f(x, u(x), v(x)) \, dx \leq \liminf_{n \to \infty} \int_D \mathcal{Q}_A f(x, u_n(x), v_n(x)) \, dx
$$

(1.5)

whenever $u_n \in L^p(\Omega; \mathbb{R}^m)$, $v_n \in L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}$, $u_n \to u$ in $L^p(D; \mathbb{R}^m)$, $v_n \to v$ in $L^q(D; \mathbb{R}^d)$. In particular $\mathcal{Q}_A f$ is $\mathcal{A}$-quasiconvex if $f$ is continuous and

$$
\frac{1}{C} |v|^q - C \leq f(v) \leq C(1 + |v|^q)
$$
for some $C > 0$, and all $v \in \mathbb{R}^d$ (see the proof of Corollary 5.7).

The lower semicontinuity result (1.5) is not covered by Theorem 3.7 in [22], where it is assumed that the integrand be $\mathcal{A}$-quasiconvex and continuous in the $v$ variable. However, as remarked in [22], in the realm of general $\mathcal{A}$-quasiconvexity the function $Q_{\mathcal{A}}f(x, u, \cdot)$ may not be continuous, even if $f(x, u, \cdot)$ is. Indeed in the degenerate case $\ker \mathcal{A} = \{0\}$ all functions are $\mathcal{A}$-quasiconvex. Also, when $N = 1$, $d = 2$, and $v = (v_1, v_2)$, consider

$$\mathcal{A}v := \begin{pmatrix} 0 & 1 \\ v_1 & v_2 \end{pmatrix}.$$ 

Then for $w \in \mathbb{R}$

$$\mathcal{A}w = \begin{pmatrix} 0 & w \end{pmatrix}$$

and thus when $|w| = 1$ the matrix $\mathcal{A}w$ has constant rank 1. For any given function $f(v)$ the $\mathcal{A}$-quasiconvex envelope of $f$ is obtained by convexification in the first component, so that by considering e.g. (cf. [22], [28])

$$f_1(v) := e^{-|v_1|v_2^2}, \quad f_2(v) := (1 + |v_1|^{1/2})$$

one gets

$$Q_{\mathcal{A}}f_1(v) = \begin{cases} 0 & \text{if } v_2 \neq 0 \\ 1 & \text{if } v_2 = 0, \end{cases} \quad Q_{\mathcal{A}}f_2(v) = \begin{cases} (1 + |v_1|^{1/2}) & \text{if } |v_2| \geq 1 \\ 1 & \text{if } |v_2| < 1. \end{cases}$$

(iii) The continuity of $f$ with respect to $v$ is essential to ensure the representation of $\mathcal{F}$ provided in Theorem 1.1, in contrast with the case where $\mathcal{A}v = 0$ if and only if $\text{curl} \ v = 0$. In fact, if $f : \mathbb{R}^{n \times N} \to [0, \infty)$ is a Borel function satisfying the growth condition

$$0 \leq f(v) \leq C(1 + |v|^p)$$

for $C > 0$, $1 \leq q < \infty$, $v \in \mathbb{R}^{n \times N}$, then it can be shown easily that

$$\mathcal{F}(w; D) = \int_D Qf(\nabla w(x)) \, dx \quad (1.6)$$

for all $D \in \mathcal{O}(\Omega)$, $w \in W^{1,q}(\Omega; \mathbb{R}^n)$, where $Qf$ is the quasiconvex envelope of $f$. Indeed, $Qf$ is a (continuous) quasiconvex function satisfying (H) (see [18], [8] Theorem 4.3); therefore by Theorem 1.1

$$w \mapsto \int_D Qf(\nabla w(x)) \, dx$$

is $W^{1,q}$-sequentially weakly lower semicontinuous, and so

$$\int_D Qf(\nabla w(x)) \, dx \leq \mathcal{F}(w; D).$$

Conversely, under hypothesis (H) it is known that $\mathcal{F}(v; \cdot)$ admits an integral representation (see Theorem 9.1 in [10], Theorem 20.1 in [15])

$$\mathcal{F}(w; D) = \int_D \varphi(\nabla w(x)) \, dx,$$

where $\varphi$ is a quasiconvex function, and $\varphi(v) \leq f(v)$ for all $v \in \mathbb{R}^{n \times N}$. Hence $\varphi \leq Qf$ and we conclude that (1.6) holds.

For general constant-rank operators $\mathcal{A}$, and if $f$ is not continuous with respect to $v$, it may happen that $\mathcal{F}_0((u, v) ; \cdot)$ is not even the trace of a Radon measure in $\mathcal{O}(\Omega)$ and thus (1.3) fails.
As an example, consider $d = 2$, $N = 1$, $\Omega := (0, 1)$, $v = (v_1, v_2)$, and let $A(v) = 0$ if and only if $v_2 = 0$ as in (ii) above. Let

$$f(v) := \begin{cases} (v_1 - 1)^2 + v_2^2, & \text{if } v_2 \in \mathbb{Q} \\ (v_1 + 1)^2 + v_2^2, & \text{if } v_2 \notin \mathbb{Q} \end{cases}$$

Although $f$ satisfies a quadratic growth condition of the type $(H)$, and $(A_3)$ holds with $q = 2$, it is easy to see that for all intervals $(a, b) \subset (0, 1)$,

$$\mathcal{F}_0((u, v); (a, b)) = \mathcal{F}_0(v; (a, b)) = \min \left\{ \int_a^b ((v_1 - 1)^2 + v_2^2) dx, \int_a^b ((v_1 + 1)^2 + v_2^2) dx \right\}$$

which is not the trace of a Radon measure on $\mathcal{O}(\Omega)$. On the other hand, it may be shown that (see the Appendix below for a proof)

$$\mathcal{F}((u, v); (a, b)) = \mathcal{F}(v; (a, b)) = \int_a^b (\psi^{**}(v_1) + v_2^2) \, dx,$$

where $\psi^{**}(v_1)$ is the convex envelope of

$$\psi(v_1) := \min \{ (v_1 - 1)^2, (v_1 + 1)^2 \}.$$

(iv) Using the growth condition $(H)$, a mollification argument, and the linearity of $A$, it can be shown that (see Remark 3.3 in [22])

$$Q_A f(x, u, v) = \inf \left\{ \int_Q f(x, u, v + w(y)) \, dy : w \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A, \int_Q w(y) \, dy = 0 \right\}.$$

We write $w \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A$ when $w \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ and $Aw = 0$ in $W^{-1, q}(Q; \mathbb{R}^d)$.

(v) We may also treat the cases $q = 1, \infty$ and $p = \infty$. See Theorem 3.6 below.

The proof of Theorem 1.1 relies heavily on the use of Young measures (see [5, 40]). However, instead of applying directly the arguments of Fonseca and Müller [22] (based on Balder’s [4] and Kristensen’s [26] approach in the curl-free case), we use these together with the blow-up method introduced by Fonseca and Müller in [20].

Although in Theorem 1.1 the functions $u$ and $v$ are not related to each other, the arguments of the proof work equally well when $u$ and $v$ are not independent. Indeed as a corollary, we can prove the following two theorems:

**Theorem 1.3.** Let $1 \leq p \leq \infty$, $s \in \mathbb{N}$, and suppose that $f : \Omega \times E^d_{[s-1]} \times E^d_s \to [0, \infty)$ is a Carathéodory function satisfying

$$0 \leq f(x, u, v) \leq C (1 + |u|^p + |v|^p), \quad 1 \leq p < \infty,$$

for a.e. $x \in \Omega$ and all $(u, v) \in E^d_{[s-1]} \times E^d_s$, where $C > 0$, and

$$f \in L^\infty_{\text{loc}}(\overline{\Omega} \times E^d_{[s-1]} \times E^d_s; [0, \infty)) \quad \text{if } p = \infty.$$

Then for every $u \in W^{s,p}(\Omega; \mathbb{R}^n)$ we have

$$\int_{\Omega} \mathcal{Q} f(x, u, \ldots, \nabla^s u) \, dx = \inf \left\{ \liminf_{k \to \infty} \int_{\Omega} f(x, u_k, \ldots, \nabla^s u_k) \, dx : \{u_k\} \subset W^{s,p}(\Omega; \mathbb{R}^n), \quad u_k \rightharpoonup u \text{ in } W^{s,p}(\Omega; \mathbb{R}^n) \quad (\overset{*}{\text{if } p = \infty}) \right\}.$$
where, for a.e. \( x \in \Omega \) and all \((u, v) \in E^p_{[s-1]} \times E^n_u\),

\[
Q^s f(x, u, v) := \inf \left\{ \int_Q f(x, u, v + \nabla^s w(y)) \, dy : w \in C^\infty_{1, \text{per}}(\mathbb{R}^N; \mathbb{R}^n) \right\},
\]

Remarks 1.4. (i) Here \( E^n_u \) stands for the space of \( n \)-tuples of symmetric \( s \)-linear maps on \( \mathbb{R}^N \),

\[
E^\infty_{[s-1]} := \mathbb{R}^n \times E^n_1 \times \cdots \times E^n_{s-1},
\]

and

\[
\nabla^l u := \left( \frac{\partial^l u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \right), \quad l \geq 1.
\]

(ii) When \( s = 1 \) we recover classical relaxation results (see e.g. the work of Acerbi and Fusco [1], Dacorogna [13], Marcellini and Sbordone [28] and the references contained therein).

When \( s > 1 \) lower semicontinuity results related to Theorem 1.3 are due to Meyers [29], Fusco [23] and Guidorzi and Poggioleti [25], while we are not aware of any integral representation formula for the relaxed energy, when the integrand depends on the full set of variables, that is \( f = f(x, u, \ldots, \nabla^s u) \). This is due to the fact that classical truncation methods for \( s = 1 \) cannot be extended in a simple way to truncate higher order derivatives. The results of Fonseca and Müller (see the proof of Lemma 2.15 in [22]), where the truncation is only on the highest order derivative \( \nabla^s u \), and Corollary 3.2 below, allows us to overcome this difficulty. Note however that this technique relies heavily on \( p \)-equi-integrability, and thus cannot work in the case \( p = 1 \), if one replaces weak convergence in \( W^{s,1}(\Omega; \mathbb{R}^n) \) with the natural convergence, which is strong convergence in \( W^{s-1,1}(\Omega; \mathbb{R}^n) \). In this context, a relaxation result has been given by Amar and De Cicco [2], but only when \( f = f(\nabla^s u) \), so that truncation is not needed. The general case where \( f \) depends also on lower order derivatives has been addressed by Fonseca, Leoni, Malý and Paroni [19].

Theorem 1.5. Let \( 1 \leq p \leq \infty \), let \( \Omega \subset \mathbb{R}^N \) be an open, bounded, connected set, and suppose that \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty) \) is a Carathéodory function satisfying

\[
0 \leq f(x, u, v) \leq C \left( 1 + |u|^p + |v|^p \right), \quad 1 \leq p < \infty,
\]

for a.e. \( x \in \Omega \) and all \((u, v) \in \mathbb{R}^N \times \mathbb{R}^N \), where \( C > 0 \), and

\[
f \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R}^N \times \mathbb{R}^N; [0, \infty)) \quad \text{if } p = \infty.
\]

Then for every \( u \in W^{1,p}(\Omega; \mathbb{R}^N) \) such that \( \text{div } u = 0 \), we have

\[
\int_\Omega \tilde{f}(x, u(x), \nabla u(x)) \, dx = \inf \left\{ \liminf_{n \to \infty} \int_\Omega f(x, u_n(x), \nabla u_n(x)) \, dx : \{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^N), \right. \left. \text{div } u_n = 0, \quad u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) \quad caller{\,} \right\}.
\]

(1.7)

where, for a.e. \( x \in \Omega \) and all \((u, v) \in \mathbb{R}^N \times \mathbb{R}^N \),

\[
\tilde{f}(x, u, v) := \inf \left\{ \int_Q f(x, u, v + \nabla w(y)) \, dy : w \in C^\infty_{1, \text{per}}(\mathbb{R}^N; \mathbb{R}^N), \text{div } w = 0 \right\}.
\]
Remark 1.6. To the authors’ knowledge, this result is new in this generality (for a different proof, with additional smoothness assumptions, see [9]). A related problem was addressed by Dal Maso, Defranceschi and Vitali in [16], where it was shown that the $\Gamma$-limit of a family of functionals of the type (1.7) may be non local if $(H)$ is violated.

In the second part of the paper we present (\$\Gamma$-convergence) homogenization results for periodic integrands in the context of $\mathcal{A}$-quasiconvexity. Let $\varepsilon > 0$ and $1 < q < \infty$, and consider a family of functionals

$$\mathcal{F}_\varepsilon : (L^q(\Omega; \mathbb{R}^d) \cap \text{ker } \mathcal{A}) \times \mathcal{O}(\Omega) \to [0, \infty)$$

defined by

$$\mathcal{F}_\varepsilon (v; D) := \int_D f \left( \frac{x}{\varepsilon}, v(x) \right) \, dx,$$

where the density $f$ satisfies the following hypotheses:

$(A_1)$ $f : \mathbb{R}^N \times \mathbb{R}^d \to [0, \infty)$ is a continuous function, $Q$-periodic in the first argument, that is $f(x + e_i, v) = f(x, v)$ for every $i = 1, \ldots, N$, where $e_i$ are the elements of the canonical basis of $\mathbb{R}^N$;

$(A_2)$ there exists $C > 0$ such that

$$0 \leq f(x, v) \leq C(1 + |v|^q)$$

for all $(x, v) \in \mathbb{R}^N \times \mathbb{R}^d$;

$(A_3)$ there exists $C > 0$ such that

$$f(x, v) \geq \frac{1}{C} |v|^q - C$$

for all $(x, v) \in \mathbb{R}^N \times \mathbb{R}^d$.

Let $\varepsilon_n \to 0^+$. We say that a functional

$$\mathcal{J} : (L^q(\Omega; \mathbb{R}^d) \cap \text{ker } \mathcal{A}) \times \mathcal{O}(\Omega) \to [0, +\infty]$$

is the $\Gamma - \liminf$ (resp. $\Gamma - \limsup$) of the sequence of functionals $\{\mathcal{F}_{\varepsilon_n}\}$ with respect to the weak convergence in $L^q(\Omega; \mathbb{R}^d)$ if for every $v \in L^q(\Omega; \mathbb{R}^d) \cap \text{ker } \mathcal{A}$

$$\mathcal{J} (v; \Omega) = \inf \left\{ \liminf_{n \to \infty} \text{ (resp. lim sup) } \mathcal{F}_{\varepsilon_n} (v_n; \Omega) : v_n \in L^q(\Omega; \mathbb{R}^d) \cap \text{ker } \mathcal{A}, \right.$$

$$v_n \rightharpoonup v \text{ in } L^q(\Omega; \mathbb{R}^d) \left. \right\},$$

and we write

$$\mathcal{J} = \Gamma - \liminf_{n \to \infty} \mathcal{F}_{\varepsilon_n} \left( \text{ resp. } \mathcal{J} = \Gamma - \limsup_{n \to \infty} \mathcal{F}_{\varepsilon_n} \right).$$

When finite energy sequences are $L^q$-equibounded then the infimum in the definition of $\Gamma - \liminf$ (resp. $\Gamma - \limsup$) is attained. We say that the sequence $\{\mathcal{F}_{\varepsilon_n}\}$ $\Gamma$-converges to $\mathcal{J}$ if the $\Gamma - \liminf$ and $\Gamma - \limsup$ coincide, and we write

$$\mathcal{J} = \Gamma - \lim_{n \to \infty} \mathcal{F}_{\varepsilon_n},$$

The functional $\mathcal{J}$ is said to be the $\Gamma - \liminf$ (resp. $\Gamma - \limsup$) of the family of functionals $\{\mathcal{F}_\varepsilon\}$ with respect to the weak convergence in $L^q(\Omega; \mathbb{R}^d)$ if for every sequence $\varepsilon_n \to 0^+$ we have that

$$\mathcal{J} = \Gamma - \liminf_{n \to \infty} \mathcal{F}_{\varepsilon_n} \left( \text{ resp. } \mathcal{J} = \Gamma - \limsup_{n \to \infty} \mathcal{F}_{\varepsilon_n} \right),$$

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and we write

\[ \mathcal{J} = \Gamma - \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon \quad \text{(resp. } \mathcal{J} = \Gamma - \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon) . \]

Finally, we say that $\mathcal{J}$ is the $\Gamma$-limit of the family of functionals $\{\mathcal{F}_\varepsilon\}$, and we write

\[ \mathcal{J} = \Gamma - \lim_{n \to \infty} \mathcal{F}_{\varepsilon_n}, \]

if $\Gamma - \liminf$ and $\Gamma - \limsup$ coincide.

In the sequel we will also consider functionals $\mathcal{J}$ given by (1.8) where we replace the weak convergence $v_n \rightharpoonup v$ with the convergence $v_n \to v$ with respect to some metric $d$. In order to highlight this dependence on the metric $d$ these functionals will be denoted as

\[ \mathcal{J} = \Gamma(d) - \liminf_{n \to \infty} \mathcal{F}_{\varepsilon_n} \quad \text{(resp. } \mathcal{J} = \Gamma(d) - \limsup_{n \to \infty} \mathcal{F}_{\varepsilon_n}), \]

as it is customary (see [15, 10]).

**Theorem 1.7.** Under hypotheses $(A_1) - (A_2)$ and the constant-rank hypothesis (1.1),

\[ \mathcal{F}_{\text{hom}} = \Gamma - \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon, \]

where

\[ \mathcal{F}_{\text{hom}}(v; D) := \int_D f_{\text{hom}}(v) \, dx \]

for all $v \in L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}$ and $D \in \mathcal{O}(\Omega)$, and

\[ f_{\text{hom}}(v) := \inf_{k \in \mathbb{N}} \frac{1}{k} \inf \left\{ \int_{kQ} f(x, v + w(x)) \, dx : \ w \in L^q_{k\text{-per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A}, \ \int_{kQ} w(x) \, dx = 0 \right\} \]

(1.9)

for all $v \in \mathbb{R}^d$. Moreover, if $(A_3)$ holds then

\[ \mathcal{F}_{\text{hom}} = \Gamma - \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon. \]

For the definition of the space $L^q_{k\text{-per}}(\mathbb{R}^N; \mathbb{R}^d)$, we direct the reader to Section 2.

**Remarks 1.8.** (i) Using the growth condition $(A_2)$, a mollification argument, and the linearity of $\mathcal{A}$, it can be shown that

\[ f_{\text{hom}}(v) = \inf_{k \in \mathbb{N}} \frac{1}{k} \inf \left\{ \int_{kQ} f(x, v + w(x)) \, dx : \ w \in L^q_{k\text{-per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A}, \ \int_{kQ} w(x) \, dx = 0 \right\}. \]

See also Corollary 5.7 below.

(ii) When $f$ satisfies the $q$-Lipschitz condition

\[ |f(x, v_1) - f(x, v_2)| \leq C(|v_1|^q - 1 + |v_2|^q - 1) |v_1 - v_2| \]

(1.10)

for all $x \in \mathbb{R}^N$, $v_1, v_2 \in \mathbb{R}^d$, and for some $C > 0$, then the continuity of $f(\cdot, v)$ can be weakened to measurability, namely $f$ can be assumed to be simply Carathéodory. Note that (1.10) is not restrictive when $\mathcal{A} = \text{curl}$, that is when $v = \nabla u$ for some $u \in W^{1,q}(\Omega; \mathbb{R}^m)$, $d = N \times m$. Indeed, in this case in the definition of $\Gamma$-convergence we may replace the weak convergence of the
gradients in $L^p(\Omega; \mathbb{R}^d)$ with the strong convergence in $L^q(\Omega; \mathbb{R}^m)$ of the potentials normalized to have zero average over $\Omega$, and thus
\[
\Gamma - \lim_{\varepsilon \to 0} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \nabla u(x) \right) \, dx = \Gamma(\mathcal{L}(\Omega; \mathbb{R}^m)) - \lim_{\varepsilon \to 0} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \nabla u(x) \right) \, dx
\]
by Proposition 7.13 in [10]. As shown in [27], if $f(x, v)$ is a Borel function which satisfies the growth condition ($A_2$) then its quasiconvex envelope $Qf$ satisfies (1.10).

A similar argument fails for general $A$-quasiconvexity, since the function $Q_A f(x, \cdot)$ may not even be continuous, see Remark 1.2 (i) above.

In Section 2 we collect preliminary results on Young measures and $\Gamma$-convergence. The general relaxation results (see Theorem 1.1 and its extension Theorem 3.6) are proved in Section 3, and Section 4 is devoted to the applications of the general relaxation principle to Theorems 1.3 and 1.5. Finally, in Section 5 we address homogenization of functionals of $A$-constrained vector fields.

2 Preliminaries

We start with some notation. Here $\Omega$ is an open, bounded subset of $\mathbb{R}^N$, $\mathcal{L}^N$ is the $N$ dimensional Lebesgue measure, $S^{N-1} := \{ x \in \mathbb{R}^N : |x| = 1 \}$ is the unit sphere, and $Q := (-1/2, 1/2)^N$ the unit cube centered at the origin. We set $Q(x_0, \varepsilon) := x_0 + \varepsilon Q$ for $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$. A function $w \in L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^d)$ is said to be $Q$-periodic if $w(x + e_i) = w(x)$ for a.e. all $x \in \mathbb{R}^N$ and every $i = 1, \ldots, N$, where $(e_1, \ldots, e_N)$ is the canonical basis of $\mathbb{R}^N$. We write $w \in L^1_{Q, \text{per}}(\mathbb{R}^N; \mathbb{R}^d)$. More generally, $w \in L^1_{kQ, \text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ is said to be $kQ$-periodic, $k \in \mathbb{N}$, if $w(k \cdot)$ is $Q$-periodic. We write $w \in L^1_{kQ, \text{per}}(\mathbb{R}^N; \mathbb{R}^d)$. Also $C^\infty_{Q, \text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ will stand for the space of $Q$-periodic functions in $C^\infty(\mathbb{R}^N; \mathbb{R}^d)$.

We recall briefly some facts about Young measures which will be useful in the sequel (see e.g. [5], [33]). If $D$ is an open set (not necessarily bounded), we denote by $C_c(D; \mathbb{R}^d)$ the set of continuous functions with compact support in $D$, endowed with the supremum norm. The dual of the closure of $C_c(D; \mathbb{R}^d)$ may be identified with the set of $\mathbb{R}^d$-valued Radon measures with finite mass $\mathcal{M}(D; \mathbb{R}^d)$, through the duality
\[
\langle \nu, f \rangle := \int_D f(y) \, d\nu(y), \quad \nu \in \mathcal{M}(D; \mathbb{R}^d), \quad f \in C_c(D; \mathbb{R}^d).
\]
A map $\nu : \Omega \to \mathcal{M}(D; \mathbb{R}^d)$ is said to be weak-* measurable if $x \mapsto \langle \nu_x, f \rangle$ are measurable for all $f \in C_c(D; \mathbb{R}^d)$.

The following result is a corollary of the Fundamental Theorem on Young Measures (see [5, 7, 34])

**Theorem 2.1.** Let $z_n : \Omega \to \mathbb{R}^d$ be measurable functions such that
\[
\sup_{n \in \mathbb{N}} \int_{\Omega} |z_n|^q \, dx < \infty,
\]
for some $q > 0$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and a weak-* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ such that
(i) $\nu_x \geq 0$, $|\nu_x|_{\mathcal{M}} = \int_{\mathbb{R}^d} d\nu_x = 1$ for a.e. $x \in \Omega$;
(ii) if $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a normal function bounded from below then
\[
\liminf_{k \to \infty} \int_{\Omega} f(x, z_{n_k}(x)) \, dx \geq \int_{\Omega} \bar{f}(x) \, dx < \infty,
\]
where
\[ f(x) := \langle \tau_x, f(x, \cdot) \rangle = \int_{\mathbb{R}^d} f(x, y) \, d\nu_x(y); \]

(iii) for any Carathéodory function \( f : \Omega \times \mathbb{R}^d \to \mathbb{R} \) bounded from below one has
\[ \lim_{k \to \infty} \int \Omega f(x, z_{n_k}(x)) \, dx = \int \Omega f(x) \, dx < \infty \]
if and only if \( \{ f(\cdot, z_{n_k}(\cdot)) \} \) is equi-integrable.

The map \( \nu : \Omega \to \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d) \) is called the Young measure generated by the sequence \( \{ z_{n_k} \} \).

**Proposition 2.2.** If \( \{ z_n \} \) generates a Young measure \( \nu \) and \( \nu_n \to 0 \) in measure, then \( \{ z_n + v_n \} \)
still generates the Young measure \( \nu \).

If \( 1 < q \leq \infty \) then \( W^{-1,q}(\Omega; \mathbb{R}^d) \) is the dual of \( W_0^{1,q'}(\Omega; \mathbb{R}^d) \), where \( q' \) is the Hölder conjugate exponent of \( q \), that is \( 1/q' + 1/q = 1 \). It is well known that \( F \in W^{-1,q}(\Omega; \mathbb{R}^d) \) if and only if there exist \( g_1, \ldots, g_N \in L^q(\Omega; \mathbb{R}^d) \) such that
\[ \langle F, w \rangle = \sum_{i=1}^N \int \Omega g_i \cdot \frac{\partial w}{\partial x_i} \, dx \quad \text{for all } w \in W_0^{1,q}(\Omega; \mathbb{R}^d). \]

Consider a collection of linear operators \( A^{(i)} : \mathbb{R}^d \to \mathbb{R}^d \), \( i = 1, \ldots, N \), and define the differential operator
\[ A : L^q(\Omega; \mathbb{R}^d) \to W^{-1,q}(\Omega; \mathbb{R}^d) \]
\[ \nu \mapsto A\nu \]
as follows:
\[ \langle A\nu, w \rangle := \sum_{i=1}^N A^{(i)} \frac{\partial \nu}{\partial x_i}, w \rangle = -\sum_{i=1}^N \int \Omega A^{(i)} \frac{\partial \nu}{\partial x_i} \, dx \quad \text{for all } w \in W_0^{1,q}(\Omega; \mathbb{R}^d). \]

Even though the operator \( A \) so defined depends on \( \Omega \), we will omit reference to the underlying domain whenever it is clear from the context. In particular, if \( \nu \in L^q_{1, \text{per}}(\mathbb{R}^N; \mathbb{R}^d) \) then we will say that \( \nu \in \ker A \) if \( A\nu = 0 \) in \( W^{-1,q}(Q; \mathbb{R}^d) \).

Throughout the paper we assume that \( A \) satisfies the constant-rank property (1.1).

The following proposition is due to Fonseca and Müller [22].

**Proposition 2.3.** (i) \( 1 < q < +\infty \) \( \{ V_n \} \) be a bounded sequence in \( L^q(\Omega; \mathbb{R}^d) \) such that \( AV_n \to 0 \) in \( W^{-1,q}(\Omega; \mathbb{R}^d) \), \( V_n \to V \) in \( L^q(\Omega; \mathbb{R}^d) \), and assume that \( \{ V_n \} \) generates a Young measure \( \nu \). Then there exists a \( q \)-equi-integrable sequence \( \{ v_n \} \subset L^s(\Omega; \mathbb{R}^d) \cap \ker A \) such that
\[ \int_{\Omega} v_n \, dx = \int_{\Omega} V \, dx, \quad \| v_n - V_n \|_{L^s(\Omega)} \to 0 \quad \text{for all } 1 \leq s < q, \]
and, in particular, \( \{ v_n \} \) still generates \( \nu \). Moreover, if \( \Omega = Q \) then \( v_n - V \in L_{1, \text{per}}^q(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A \).

(ii) \( q = 1 \) \( \{ V_n \} \) be a sequence converging weakly in \( L^1(\Omega; \mathbb{R}^d) \) to a function \( V \), \( AV_n \to 0 \) in \( W^{-1,r}(\Omega; \mathbb{R}^d) \) for some \( r \in (1, N/(N-1)) \), and assume that \( \{ V_n \} \) generates a Young measure \( \nu \). Then there exists an equi-integrable sequence \( \{ v_n \} \subset L^1(\Omega; \mathbb{R}^d) \cap \ker A \) such that
\[ \int_{\Omega} v_n \, dx = \int_{\Omega} V \, dx, \quad \| v_n - V_n \|_{L^1(\Omega)} \to 0, \]
and, in particular, \( \{ v_n \} \) still generates \( \nu \). Moreover, if \( \Omega = Q \) then \( v_n - V \in L_{1, \text{per}}^q(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A \).
(iii) \((q = +\infty)\) Let \(\{V_n\}\) be a sequence that satisfies \(V_n \xrightarrow{\ast} V\) in \(L^\infty(\Omega; \mathbb{R}^d)\), \(AV_n \rightharpoonup 0\) in \(L^r(\Omega)\) for some \(r > N\), and assume that \(\{V_n\}\) generates a Young measure \(\nu\). Then there exists a sequence \(\{v_n\} \subset L^\infty(\Omega; \mathbb{R}^d) \cap \ker A\) such that
\[
\int \Omega v_n \, dx = \int \Omega V \, dx, \quad ||v_n - V_n||_{L^\infty(\Omega)} \to 0,
\]
and, in particular, \(\{v_n\}\) still generates \(\nu\). Moreover, if \(\Omega = Q\) then \(v_n - V \in L^\infty_{\text{top}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A\).

In the second part of the paper we will need the following classical results from \(\Gamma\)-convergence. For a proof see [10].

**Proposition 2.4.** Let \((X, d)\) be a separable metric space and let \(f_n : X \to [-\infty, \infty]\). Then
(i) there exists an increasing sequence of integers \(\{n_k\}\) such that
\[
\Gamma(d) \leftarrow \lim_{k \to \infty} f_{n_k}(x) \quad \text{exists for all } x \in X.
\]
(ii) Moreover
\[
f_\infty = \Gamma(d) \leftarrow \lim_{n \to \infty} f_n
\]
if and only if for every subsequence \(\{f_{n_k}\}\) there exists a further subsequence \(\{f_{n_{k_l}}\}\) which \(\Gamma(d)\)-converges to \(f_\infty\).

## 3 Relaxation

In this section we prove Theorem 1.1 and its generalization to the case where \(q \in \{1, \infty\}\) and \(p = \infty\) (see Theorem 3.6).

**Lemma 3.1.** Let \(f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, \infty)\) be a Carathéodory function satisfying (H), with \(1 \leq p < \infty\) and \(1 < q < \infty\). Let \((u, v) \in L^p(D; \mathbb{R}^m) \times \left(L^q(D; \mathbb{R}^d) \cap \ker A\right)\), where \(D \in \mathcal{O}(\Omega)\), and consider a sequence of functions \(\{(u_k, \hat{v}_k)\} \subset L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d)\) such that
\[
\begin{align*}
    u_k &\to u \quad \text{in} \quad L^p(D; \mathbb{R}^m), \quad \hat{v}_k \to v \quad \text{in} \quad L^q(D; \mathbb{R}^d) \\
    A\hat{v}_k &\to 0 \quad \text{in} \quad W^{-1,q}(D; \mathbb{R}^d).
\end{align*}
\]

Then we can find a \(q\)-equi-integrable sequence \(\{v_k\} \subset L^q(D; \mathbb{R}^d) \cap \ker A\) such that
\[
v_k \to v \quad \text{in} \quad L^q(D; \mathbb{R}^d), \quad \int_D v_k \, dx = \int_D v \, dx,
\]
and
\[
\liminf_{k \to \infty} \int_D f(x, u(x), v_k(x)) \, dx \leq \liminf_{k \to \infty} \int_D f(x, u_k(x), \hat{v}_k(x)) \, dx.
\]

**Proof of Lemma 3.1.** Consider a subsequence \(\{(u_n, \hat{v}_n)\}\) of \(\{(u_k, \hat{v}_k)\}\) such that
\[
\lim_{n \to \infty} \int_D f(x, u_n(x), \hat{v}_n(x)) \, dx = \liminf_{k \to \infty} \int_D f(x, u_k(x), \hat{v}_k(x)) \, dx
\]
and \(\{(u_n, \hat{v}_n)\}\) generates the Young measure \(\{\delta_{u(x)} \otimes \nu_x\}_{x \in D}\). For \(i \in \mathbb{N}\) let
\[
F_i := \left\{ x \in D : \text{dist}(x, \partial D) < \frac{1}{i} \right\},
\]
and consider cut-off functions $\theta_i$ with compact support in $D$ and such that $\theta_i \equiv 1$ in $D \setminus F_i$. Let $w_{i,n} := \theta_i(\hat{w}_i - v) \in L^q(D; \mathbb{R}^d)$ and fix $\varphi \in L^q(D; \mathbb{R}^d)$, where $q'$ is the Hölder conjugate exponent of $q$. Then

$$
\lim_{i \to \infty} \lim_{n \to \infty} \int_D \varphi(x) w_{i,n}(x) \, dx = \lim_{i \to \infty} \lim_{n \to \infty} \int_D \varphi(x) \theta_i(x)(\hat{w}_i(x) - v(x)) \, dx = 0, \quad (3.2)
$$

where we have used the fact that $\hat{w}_i \to v$ in $L^q(D; \mathbb{R}^d)$. Hence $w_{i,n} \to 0$ in $L^q(D; \mathbb{R}^d)$ as $n \to \infty$ and $i \to \infty$. Moreover, in view of the compact embedding

$$
L^q(D; \mathbb{R}^d) \hookrightarrow W^{-1,q}(D; \mathbb{R}^d)
$$

and the assumption that $A\hat{w}_i \to 0$ in $W^{-1,q}(D; \mathbb{R}^d)$, we have that

$$
\lim_{i \to \infty} \lim_{n \to \infty} A w_{i,n} = 0 \quad \text{in} \quad W^{-1,q}(D; \mathbb{R}^d).
$$

Let $G$ be a countable dense subset of $L^q(D; \mathbb{R}^d)$. By means of a diagonalization process we obtain subsequences \{u_i := u_{i,n}\} and \{\hat{w}_i := w_{i,n} - \theta_i(\hat{w}_i - v)\} such that $\|u_i - u\|_{L^r} \to 0$, (3.2) holds for each $\varphi \in G$, and

$$
A \hat{w}_i \to 0 \quad \text{in} \quad W^{-1,q}(D; \mathbb{R}^d).
$$

Hence $\hat{w}_i \to 0$ in $L^q(D; \mathbb{R}^d)$, by the density of $G$ in $L^q(D; \mathbb{R}^d)$. By Proposition 2.3(i) there exists a $q$-equi-integrable sequence \{w_i\} $\subset L^q(D; \mathbb{R}^d) \cap \ker A$ such that $w_i \to 0$ in $L^q(D; \mathbb{R}^d)$, and

$$
\int_D w_i \, dx = 0, \quad \|\hat{w}_i - w_i\|_{L^r(D)} \to 0 \quad \text{for all} \ 1 \leq s < q.
$$

Set $v_i := v + w_i$. Then $\int_D v_i \, dx = \int_D v \, dx$, $v_i \to v$ in $L^q(D; \mathbb{R}^d)$. By Hölder’s inequality and by (3.3), for $1 \leq s < q$

$$
\|\hat{v}_n - v_i\|_{L^q(D)} \leq \|\hat{v}_n - v - \hat{w}_i\|_{L^q(D)} + \|\hat{w}_i - w_i\|_{L^q(D)} \\
\leq \|(1 - \theta_i)(\hat{v}_n - v)\|_{L^q(D)} + \|\hat{w}_i - w_i\|_{L^q(D)} \\
\leq \|\hat{v}_n - v\|_{L^q(D)} + \|\hat{w}_i - w_i\|_{L^q(D)} \to 0
$$

as $i \to \infty$ and where $r := (q - s)/sq$. By (3.4) and Proposition 2.2, the two sequences

$$
\{(u(x), v_i(x))\} \quad \text{and} \quad \{(u_i(x), \hat{v}_i(x))\}
$$

generate the same Young measure $\{\delta_{u(x)} \otimes \nu_x\}_{x \in D}$. Hence by Theorem 2.1 (ii) and (iii)

$$
\lim_{i \to \infty} \int_D f(x, u(x), v_i(x)) \, dx = \int_D \int_{\mathbb{R}^d} f(x, u(x), V) \, d\nu_x(V) \, dx
$$

$$
\leq \lim_{i \to \infty} \inf \int_D f(x, u(x), \hat{v}_i(x)) \, dx = \lim_{i \to \infty} \inf \int_{k \to \infty} \int_D f(x, uk(x), \hat{v}_k(x)) \, dx,
$$

where we have used the fact that $\{f(x, u(x), v_i(x))\}$ is equi-integrable over $D$, which follows from (H) and the $q$-equi-integrability of $\{v_i\}$ over $D$. \hfill \square

It follows immediately from Lemma 3.1 that under its assumptions on $f$ it holds:

**Corollary 3.2.** For $D \in \mathcal{O}(\Omega)$ and $(u, v) \in L^p(\Omega; \mathbb{R}^m) \times (L^q(\Omega; \mathbb{R}^d) \cap \ker A)$

$$
F((u, v); D) \leq \inf \left\{ \liminf_{n \to \infty} \int_D g(x, v_n(x)) \, dx : \{v_n\} \subset L^q(D; \mathbb{R}^d) \cap \ker A \quad \text{is q-equi-integrable} \right\},
$$

where $g$ is the Carathéodory function defined by

$$
g(x, v) := f(x, u(x), v).
$$

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Note that, by (H), the function $g$ satisfies the growth condition
\begin{equation}
0 \leq g(x, v) \leq C (1 + |u(x)|^p + |v|^q)
\end{equation}
for a.e. $x \in \Omega$ and all $v \in \mathbb{R}^d$. Moreover, since $g$ is a Carathéodory function, by the Scorza-Dragoni Theorem for each $j \in \mathbb{N}$ there exists a compact set $K_j \subset \Omega$, with $|\Omega \setminus K_j| \leq 1/j$, such that $g : K_j \times \mathbb{R}^d \to [0, \infty)$ is continuous. Let $K_j^*$ be the set of Lebesgue points of $\chi_{K_j}$, and set
\begin{equation}
\omega := \bigcup_{j=1}^{\infty} (K_j \cap K_j^*) \cap L(u, v),
\end{equation}
where $L(u, v)$ is the set of Lebesgue points of $(u, v)$. Then
$$|\Omega \setminus \omega| \leq |\Omega \setminus K_j| \leq \frac{1}{j} \to 0 \text{ as } j \to \infty.$$ 

**Corollary 3.3.** Assume that $x_0 \in \omega$, let $v \in L^q(Q; \mathbb{R}^d) \cap \ker A$, and consider $r_k \to 0^+$ and a sequence of functions
$$\{\hat{v}_k\} \subset L^q(Q; \mathbb{R}^d) \cap \ker A$$
such that
$$\hat{v}_k \rightharpoonup v \text{ in } L^q(Q; \mathbb{R}^d).$$
Then we can find a $q$-equi-integrable sequence $\{w_k\} \subset L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A$ such that
$$w_k \to 0 \text{ in } L^q(Q; \mathbb{R}^d), \quad \int_Q w_k \, dx = 0,$$
and
$$\liminf_{k \to \infty} \int_Q g(x_0, v(y) + w_k(y)) \, dy \leq \liminf_{k \to \infty} \int_Q g(x_0 + r_k y, \hat{v}_k(y)) \, dy.$$

**Proof of Corollary 3.3.** We proceed as in the proof of Lemma 3.1 up to (3.4). Since the sequence $\{v_i\}$ is $q$-equi-integrable, for any $\eta > 0$ there exists $\delta > 0$ such that
\begin{equation}
\sup_i \int_D C(1 + |u(x_0)|^p + |v_i(y)|^q) \, dy < \eta
\end{equation}
for any measurable set $D \subset Q$, with $|D| < \delta$, and where $C$ is the constant given in (H). Fix $\eta > 0$ and let $\delta > 0$ be given according to (3.8). By the Biting Lemma (see [6]) we may find a further subsequence $\{\hat{v}_{n_j}\} \subset \{v_{n_j}\}$ and a set $E \subset Q$ such that $|Q \setminus E| < \delta$ and $\{\hat{v}_{n_j}\}$ is $q$-equi-integrable over $E$. Hence there exists $0 < \delta_1 < \delta$ such that
\begin{equation}
\sup_j \int_D C(1 + |u(x_0)|^p + |\hat{v}_{n_j}(y)|^q) \, dy < \eta
\end{equation}
for any measurable set $D \subset E$, with $|D| < \delta_1$. Moreover, as $\{\hat{v}_{n_j}\}, \{v_j\}$ are bounded in $L^q(Q; \mathbb{R}^d)$, we may find $L > 0$ such that
\begin{equation}
|E \setminus E_j| \leq \delta_1, \quad \text{where} \quad E_j := \{y \in E : |\hat{v}_{n_j}(y)| \leq L, |v_j(y)| \leq L\}.
\end{equation}

Note that by construction of $v_j$ and by Proposition 2.3, $v_i = v + w_i$ where $w_i \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A$. From the definition of the set $\omega$ there exists an integer $j_0$ such that $x_0 \in K_{j_0} \cap K_{j_0}^*$. Since $g : K_{j_0} \times \overline{B_d(0, L)} \to [0, \infty)$

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is uniformly continuous, there exists $\rho > 0$ such that
\begin{equation}
|g(x, v) - g(x_1, v)| \leq \eta \tag{3.11}
\end{equation}
for all $(x, v), (x_1, v) \in K_{j_n} \times \overline{B_d(0, L)}$, with $|x - x_1| \leq \rho$. By (3.10) and (3.11)
\begin{align*}
\lim_{i \to \infty} \int_Q g(x_0 + r_n y, \hat{v}_{n_j}(y)) \, dy & \geq \liminf_{j \to \infty} \int_{E_j} g(x_0 + r_n y, \hat{v}_{n_j}(y)) \, dy \\
& \geq \liminf_{j \to \infty} \frac{1}{r_{n_j}^N} \int_{(x_0 + r_n E_j) \cap K_{j_n}} g(x, \hat{v}_{n_j}((x - x_0)/r_{n_j})) \, dx \\
& \geq -\eta + \liminf_{j \to \infty} \frac{1}{r_{n_j}^N} \int_{(x_0 + r_n E_j) \cap K_{j_n}} g(x_0, \hat{v}_{n_j}((x - x_0)/r_{n_j})) \, dx.
\end{align*}
\begin{equation}
\tag{3.12}
\end{equation}
Using, once again, the fact that $|\hat{v}_{n_j}(y)| \leq L$ for $y \in E_j$, by (3.6) we have that
\begin{align*}
\frac{1}{r_{n_j}^N} \int_{(x_0 + r_n E_j) \cap K_{j_n}} g(x_0, \hat{v}_{n_j}((x - x_0)/r_{n_j})) \, dx & \leq C(1 + |u(x_0)|^p + L^q) \frac{|Q(x_0, r_{n_j}) \setminus K_{j_n}|}{r_{n_j}^N} \to 0
\end{align*}
as $j \to \infty$, because $x_0$ is a Lebesgue point of $\chi_{K_{j_n}}$. Consequently, from (3.12) we get
\begin{align*}
\lim_{i \to \infty} \int_Q g(x_0 + r_n y, \hat{v}_{n_j}(y)) \, dy & \geq -\eta + \liminf_{j \to \infty} \frac{1}{r_{n_j}^N} \int_{x_0 + r_n E_j} g(x_0, \hat{v}_{n_j}((x - x_0)/r_{n_j})) \, dx \\
& = -\eta + \liminf_{j \to \infty} \int_{E_j} g(x_0, \hat{v}_{n_j}(y)) \, dy \\
& \geq -2\eta + \liminf_{j \to \infty} \int_E g(x_0, \hat{v}_{n_j}(y)) \, dy,
\end{align*}
where we have used (3.6), (3.9) and the fact that $|E \setminus E_j| \leq \delta_1$. We may now proceed as in the previous lemma, using the Carathéodory function $h(x, v) := \chi_{E}(x)g(x_0, v)$, to obtain
\begin{align*}
\lim_{i \to \infty} \int_Q g(x_0 + r_n y, \hat{v}_{n_j}(y)) \, dy & \geq -2\eta + \liminf_{j \to \infty} \int_E g(x_0, v_j(y)) \, dy \\
& \geq -3\eta + \liminf_{j \to \infty} \int_Q g(x_0, v_j(y)) \, dy
\end{align*}
by (3.8). It now suffices to let $\eta \to 0^+$. \hfill \Box

Theorem 1.1 follows from Lemmas 3.4 and 3.5 below. We will use the notation $\mu|A$ to denote the restriction of a Radon measure $\mu$ to the Borel set $A$, i.e., $\mu|A(X) := \mu(X \cap A)$ where $X$ is an arbitrary Borel set in the domain of $\mu$.

**Lemma 3.4.** $F((u, v); \cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^N|\Omega$.

**Proof of Lemma 3.4.** As it is usual, it suffices to prove subadditivity (see e.g. [3, 21]), i.e.
\begin{equation}
F((u, v); D) \leq F((u, v); D \setminus B) + F((u, v); C)
\end{equation}
if $B \subset \subset C \subset \subset D$. Fix $\eta > 0$. By Corollary 3.2 there exist two $q$-equi-integrable sequences
\begin{equation}
\{v_k\} \subset L^q(D \setminus \overline{B}; \mathbb{R}^d) \cap \ker A, \quad \{w_k\} \subset L^q(C; \mathbb{R}^d) \cap \ker A,
\end{equation}
such that
\begin{equation}
v_k \to v \quad \text{in} \quad L^q(D \setminus \overline{B}; \mathbb{R}^d), \quad w_k \to v \quad \text{in} \quad L^q(C; \mathbb{R}^d),
\end{equation}

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and
\[
\lim_{k \to \infty} \int_{D \setminus \overline{B}} g(x, v_k(x)) \, dx \leq \mathcal{F}((u, v); D \setminus \overline{B}) + \eta,
\]
\[
\lim_{k \to \infty} \int_{C} g(x, w_k(x)) \, dx \leq \mathcal{F}((u, v); C) + \eta.
\]
Let \( \theta_j \) be smooth cut-off functions, \( \theta_j \in C_c^\infty(C; [0, 1]), \theta_j(x) = 1 \) for all \( x \in B \), and \(|\{0 < \theta_j < 1\}| \to 0 \) as \( j \to \infty \). Set
\[
\tilde{V}_{j,k} := (1 - \theta_j)v_k + \theta_jw_k.
\]
Then, for \( j \) fixed,
\[
A\tilde{V}_{j,k} = (1 - \theta_j)A v_k + \theta_jA w_k - \sum_{i=1}^{N} A^{(i)} v_k \frac{\partial \theta_j}{\partial x_i} + \sum_{i=1}^{N} A^{(i)} w_k \frac{\partial \theta_j}{\partial x_i} \to 0
\]
as \( k \to \infty \) in \( W^{-1, q}(D; \mathbb{R}^d) \) strong. Using a diagonalization procedure such as that adopted in the proof of Lemma 3.1, we get
\[
\tilde{V}_j \to v \quad \text{in} \quad L^q(D; \mathbb{R}^d), \quad A\tilde{V}_j \to 0 \quad \text{in} \quad W^{-1, q}(D; \mathbb{R}^d),
\]
where \( \tilde{V}_j := \tilde{V}_{j,k} \). By Lemma 3.1 we can find a \( q \)-equi-integrable sequence \( \{V_j\} \subset L^q(D; \mathbb{R}^d) \cap \ker A \) such that \( V_j \to v \) in \( L^q(D; \mathbb{R}^d) \) and
\[
\liminf_{j \to \infty} \int_{D} g(x, V_j(x)) \, dx \leq \liminf_{j \to \infty} \int_{D} g(x, \tilde{V}_j(x)) \, dx.
\]
Consequently, in view of Corollary 3.2
\[
\mathcal{F}((u, v); D) \leq \liminf_{j \to \infty} \int_{D} g(x, V_j(x)) \, dx \leq \liminf_{j \to \infty} \int_{D} g(x, \tilde{V}_j(x)) \, dx
\]
\[
\leq \limsup_{j \to \infty} \int_{\{\theta_j = 0\}} g(x, v_k(x)) \, dx + \limsup_{j \to \infty} \int_{\{\theta_j = 1\}} g(x, w_k(x)) \, dx
\]
\[
+ \limsup_{j \to \infty} \int_{\{0 < \theta_j < 1\}} C(1 + |u(x)|^p + |w_k(x)|^q + |v_k(x)|^r) \, dx
\]
\[
\leq 2\eta + \mathcal{F}((u, v); D \setminus \overline{B}) + \mathcal{F}((u, v); C).
\]
It suffices to let \( \eta \to 0^+ \). Finally, note that by (H) we have that
\[
\mathcal{F}((u, v), \cdot) \leq C(1 + |u|^p + |v|^q)\mathcal{L}^N|\Omega.
\]

\[\square\]

**Lemma 3.5.** For \( \mathcal{L}^N \) a.e. \( x_0 \in \Omega \) we have
\[
\frac{d\mathcal{F}((u, v); \cdot)}{d\mathcal{L}^N}(x_0) = Q_A f(x_0, u(x_0), v(x_0)).
\]

**Proof of Lemma 3.5.** Fix \( x_0 \in \omega \), where \( \omega \) is defined as in (3.7), and such that
\[
\lim_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0, r)} |u(x) - u(x_0)|^p \, dx = \lim_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0, r)} |v(x) - v(x_0)|^r \, dx = 0 \quad (3.13)
\]
and
\[
\frac{d\mathcal{F}((u, v); \cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r \to 0^+} \frac{\mathcal{F}((u, v); Q(x_0, r))}{r^N} < \infty,
\]

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where, by virtue of Lemma 3.4, we have chosen the radii \( r \to 0^+ \) such that
\[
\mathcal{F}((u, v); \partial Q(x_0, r))) = 0.
\]
By Corollary 3.2 and for \( r > 0 \) fixed, let \( \{v_{n,r}\} \subset L^q(Q(x_0, r); \mathbb{R}^d) \cap \ker A \) be such that \( v_{n,r} \to v \) in \( L^q(Q(x_0, r); \mathbb{R}^d) \) as \( n \to \infty \) and
\[
\lim_{n \to \infty} \int_{Q(x_0, r)} g(x, v_{n,r}(x)) \, dx \leq \mathcal{F}((u, v); Q(x_0, r)) + r^{N+1}.
\]
Then
\[
\frac{d\mathcal{F}((u, v); \cdot)}{dL^N}(x_0) \geq \liminf_{r \to 0^+} \lim_{n \to \infty} \frac{1}{r^N} \int_{Q(x_0,r)} g(x, v_{n,r}(x)) \, dx
\]
\[
= \liminf_{r \to 0^+} \lim_{n \to \infty} \int_{Q} g(x_0 + ry, v(x_0) + w_{n,r}(y)) \, dy
\]
where \( w_{n,r}(y) := v_{n,r}(x_0 + ry) - v(x_0) \). We claim that \( w_{n,r} \to 0 \) in \( L^q(Q; \mathbb{R}^d) \) if we first let \( n \to \infty \) and then \( r \to 0^+ \). Indeed let \( \varphi \in L^q(Q; \mathbb{R}^d) \), where \( q' \) is the Hölder conjugate exponent of \( q \). Using Hölder’s inequality and then making a change of variables, we get
\[
\left| \int_{Q} \varphi(y) w_{r,n}(y) \, dy \right| \leq \left| \int_{Q} \varphi(y)(v_{n,r}(x_0 + ry) - v(x_0 + ry)) \, dy \right|
\]
\[
+ \left| \int_{Q} \varphi(y)(v(x_0 + ry) - v(x_0)) \, dy \right|
\]
\[
\leq \left| \int_{Q} \varphi((x - x_0)/r)(v_{n,r}(x) - v(x)) \, dx \right|
\]
\[
+ \|\varphi\|_{L^{q'}(Q)} \left( \frac{1}{r^N} \int_{Q(x_0,r)} |v(x) - v(x_0)|^q \, dx \right)^{1/q}.
\]
If we now let \( n \to \infty \) the first integral tends to zero, since \( v_{n,r} \to v \) in \( L^q(Q(x_0, r); \mathbb{R}^d) \). The claim then follows by letting \( r \to 0^+ \) and by using (3.13). Diagonalize to get \( \hat{w}_k \in L^q(Q; \mathbb{R}^d) \cap \ker A \) such that \( \hat{w}_k \to 0 \) in \( L^q(Q; \mathbb{R}^d) \) and
\[
\frac{d\mathcal{F}((u, v); \cdot)}{dL^N}(x_0) \geq \lim_{k \to \infty} \int_{Q} g(x_0 + r_k y, v(x_0) + \hat{w}_k(y)) \, dy
\]
where \( r_k \to 0 \). By Corollary 3.3 there is a \( q \)-equi-integrable sequence \( \{w_k\} \subset L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A \) such that
\[
w_k \to 0 \quad \text{in} \quad L^q(Q; \mathbb{R}^d), \quad \int_Q w_k \, dy = 0,
\]
and
\[
\frac{d\mathcal{F}((u, v); \cdot)}{dL^N}(x_0) \geq \lim_{k \to \infty} \int_{Q} g(x_0 + r_k y, v(x_0) + \hat{w}_k(y)) \, dy
\]
\[
\geq \liminf_{k \to \infty} \int_{Q} f(x_0, u(x_0), v(x_0) + w_k(y)) \, dy \geq Q_A f(x_0, u(x_0), v(x_0)).
\]
To conclude the proof of the lemma it remains to show that
\[
\frac{d\mathcal{F}((u, v); \cdot)}{dL^N}(x_0) \leq Q_A f(x_0, u(x_0), v(x_0)) \quad \text{for} \quad L^N \text{ a.e.} \ x_0 \in \Omega.
\]
Fix \( \eta > 0 \) and let \( w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A \) be such that \( \int_Q w \, dy = 0 \) and
\[
\int_Q f(x_0, u(x_0), v(x_0) + w(y)) \, dy \leq Q_A f(x_0, u(x_0), v(x_0)) + \eta.
\]
For any fixed $r > 0$ set $w_{n,r}(x) := w(n(x - x_0)/r)$. Then $w_{n,r} \to 0$ in $L^q(Q(x_0, r); \mathbb{R}^d)$ as $n \to \infty$. Hence, by Corollary 3.2,

$$
\frac{d\mathcal{F}(u, v); : \cdot(x_0)}{dL^N}(x_0) = \lim_{r \to 0^+} \frac{\mathcal{F}((u, v); Q(x_0, r))}{r^N} \leq \liminf_{r \to 0^+} \liminf_{n \to \infty} \frac{1}{r^N} \int_{Q(x_0, r)} g(x, v(x) + w_{n,r}(x)) \, dx.
$$

(3.15)

Fix $L > |v(x_0)| + \|w\|_{L^\infty} + 1$, and let $j$ be such that $x_0 \in K_j \cap K_j^* \cap L(u, v)$, where we are using the notation introduced in (3.7). Since

$$
g : K_j \times \overline{B_d(0, L)} \to [0, \infty)
$$

is uniformly continuous, there exists $0 < \rho < 1$ such that

$$
|g(x, v) - g(x, v_1)| \leq \eta
$$

(3.16)

for all $(x, v), (x_1, v_1) \in K_j \times \overline{B_d(x_0, L)}$, with $|x - x_1| \leq \rho$ and $|v - v_1| \leq \rho$. Let

$$
E_{r, \rho} := \{x \in Q(x_0, r) : |v(x) - v(x_0)| \leq \rho\}.
$$

We claim that

$$
\limsup_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0, r) \backslash (E_{r, \rho} \cap K_j)} C(1 + |u(x)|^p + |v(x)|^q + \|w\|_{L^\infty}^q) \, dx = 0.
$$

(3.17)

Since $|v(x) - v(x_0)| \geq \rho$ for $x \in Q(x_0, r) \backslash E_{r, \rho}$, we have

$$
\frac{1}{r^N} \int_{Q(x_0, r) \backslash (E_{r, \rho} \cap K_j)} C(1 + |u(x)|^p + |v(x)|^q + \|w\|_{L^\infty}^q) \, dx
datax
\leq C \frac{|Q(x_0, r) \backslash (E_{r, \rho} \cap K_j)|}{r^N} + \frac{C}{r^N} \int_{Q(x_0, r)} |u(x) - u(x_0)|^p + |v(x) - v(x_0)|^q \, dx.
$$

and

$$
\frac{|Q(x_0, r) \backslash (E_{r, \rho} \cap K_j)|}{r^N} \leq \frac{|Q(x_0, r) \backslash K_j|}{r^N} + \frac{|Q(x_0, r) \backslash E_{r, \rho}|}{r^N}
$$

$$
\leq \frac{|Q(x_0, r) \backslash K_j|}{r^N} + \frac{C}{\rho^q} \frac{1}{r^N} \int_{Q(x_0, r)} |v(x) - v(x_0)|^q \, dx \to 0 \quad \text{as} \quad r \to 0^+,
$$

where we have used (3.13) and the fact that $x_0$ is a Lebesgue point of $\chi_{K_j}$. Then by (3.6), (3.16), (3.17), (3.15) and (3.14),

$$
\frac{d\mathcal{F}(u, v); : \cdot(x_0)}{dL^N}(x_0) \leq \liminf_{n \to \infty} \frac{1}{r^N} \int_{E_{r, \rho} \cap K_j} g(x, v(x) + w_{n,r}(x)) \, dx
$$

$$
+ \limsup_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0, r) \backslash (E_{r, \rho} \cap K_j)} C(1 + |u(x)|^p + |v(x)|^q + \|w\|_{L^\infty}^q) \, dx
$$

$$
\leq \eta + \liminf_{n \to \infty} \frac{1}{r^N} \int_{Q(x_0, r)} g(x_0, v(x_0) + w_{n,r}(x)) \, dx
$$

$$
= \eta + \liminf_{n \to \infty} \int_Q g(x_0, v(x_0) + w(ny)) \, dy
$$

$$
= \eta + \int_Q g(x_0, v(x_0) + w(y)) \, dy \leq 2\eta + Q_{Af}(x_0, u(x_0), v(x_0)),
$$

by virtue of the equality

$$
\liminf_{n \to \infty} \int_Q g(x_0, v(x_0) + w(ny)) \, dx = \int_Q g(x_0, v(x_0) + w(y)) \, dy,
$$

which follows from the $Q$-periodicity of the function $g(x_0, v(x_0) + w(\cdot))$. It now suffices to let $\eta \to 0^+$. □
As mentioned in the Introduction, Theorem 1.1 continues to hold when \( q \in \{1, \infty\} \) and \( p = \infty \). Indeed, let \( 1 \leq p, q \leq \infty \) and assume that

\[
(A_4) \ f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, \infty) \text{ is a Carathéodory function satisfying the following growth conditions for a.e. } x \in \Omega \text{ and all } (u, v) \in \mathbb{R}^m \times \mathbb{R}^d:
\]

\[
0 \leq f(x, u, v) \leq C (1 + |u|^p + |v|^q) \quad \text{if } 1 \leq p, q < \infty,
\]

where \( C > 0 \);

\[
0 \leq f(x, u, v) \leq a(x, u) (1 + |v|^q) \quad \text{if } p = \infty \text{ and } 1 \leq q < \infty,
\]

where \( a \in L_{\text{loc}}^\infty(\Omega \times \mathbb{R}^d; [0, \infty)) \);

\[
0 \leq f(x, u, v) \leq b(x, v) (1 + |u|^p) \quad \text{if } 1 \leq p < \infty \text{ and } q = \infty,
\]

where \( b \in L_{\text{loc}}^\infty(\Omega \times \mathbb{R}^m; [0, \infty)) \);

\[
f \in L_{\text{loc}}^\infty(\Omega \times \mathbb{R}^m \times \mathbb{R}^d; [0, \infty)) \quad \text{if } p = q = \infty.
\]

For \( D \in \mathcal{O}(\Omega) \) and \( (u, v) \in L^p(\Omega; \mathbb{R}^m) \times (L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}) \) define

\[
\mathcal{F}((u, v); D) := \inf \left\{ \liminf_{n \to \infty} F((u_n, v_n); D) : (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d), \ u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \ v_n \to v \text{ in } L^q(D; \mathbb{R}^d), \ \mathcal{A}v_n \to 0 \text{ in } W^{-1,r}(D; \mathbb{R}^d) \right\}
\]

if \( q = 1 \) and for some \( r \in (1, N/(N-1)) \); as in (1.2), we set

\[
\mathcal{F}((u, v); D) := \inf \left\{ \liminf_{n \to \infty} F((u_n, v_n); D) : (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d), \ u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \ v_n \to v \text{ in } L^q(D; \mathbb{R}^d), \ \mathcal{A}v_n \to 0 \text{ in } W^{-1,q}(D; \mathbb{R}^d) \right\}
\]

if \( 1 < q < \infty \);

\[
\mathcal{F}((u, v); D) := \inf \left\{ \liminf_{n \to \infty} F((u_n, v_n); D) : (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^\infty(D; \mathbb{R}^d), \ u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \ v_n \to v \text{ in } L^\infty(D; \mathbb{R}^d), \ \mathcal{A}v_n \to 0 \text{ in } L^r(D; \mathbb{R}^d) \right\}
\]

if \( q = \infty \) and for some \( r > N \).

We can prove the following theorem:

**Theorem 3.6.** Under condition \( (A_4) \) and the constant-rank hypothesis (1.1), for all \( D \in \mathcal{O}(\Omega) \), \( u \in L^p(\Omega; \mathbb{R}^m) \) and \( v \in L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A} \), we have

\[
\mathcal{F}((u, v); D) = \int_D Q_Af(x, u(x), v(x)) \, dx.
\]

**Proof of Theorem 3.6.**

**Step 1:** Assume first that \( 1 \leq p < \infty \) and \( q = 1 \). The proof is similar to the one of Theorem 1.1, with the exceptions that in Lemma 3.1 condition (3.1) should be replaced by

\[
u_h \to u \text{ in } L^p(D; \mathbb{R}^m), \ \hat{v}_h \to v \text{ in } L^1(D; \mathbb{R}^d), \ \mathcal{A}\hat{v}_n \to 0 \text{ in } W^{-1,r}(D; \mathbb{R}^d) \text{ for some } r \in (1, N/(N-1)),
\]
that we use the compact embedding
\[ L^1(D; \mathbb{R}^d) \hookrightarrow W^{-1,r}(D; \mathbb{R}^d), \quad r \in (0,N/(N-1)), \]
to diagonalize \( \{w_{i,n}\} \), and (3.3), (3.4) are replaced, respectively, by
\[
\int_D w_i \, dx = 0, \quad \|w_i - w_i^\dagger\|_{L^1(D)} \to 0,
\]
\[
\|\hat{w}_{i,n} - v_i\|_{L^1(D)} \leq \|\hat{w}_{i,n} - v - \hat{w}_i\|_{L^1(D)} + \|\hat{w}_i - w_i\|_{L^1(D)}
\]
\[
\leq \|(1 - \Theta_i)(\hat{w}_{i,n} - v)\|_{L^1(D)} + \|\hat{w}_i - w_i\|_{L^1(D)}
\]
\[
\leq \|\hat{w}_{i,n} - v\|_{L^1(F_i)} + \|\hat{w}_i - w_i\|_{L^1(D)} \to 0,
\]
where we have used the fact that \( \|\hat{w}_{i,n} - v\|_{L^1(F_i)} \to 0 \) as \( i \to \infty \), which is due to the equi-
integrability of the original sequence \( \{\hat{w}_k - v\} \) and the fact that \( |F_i| \to 0 \).

**Step 2:** If \( p = \infty \) and \( 1 \leq q < \infty \) then in Lemma 3.1 the only change needed is in deriving
(3.5), which now follows from the fact that, by (3.18),
\[
0 \leq f(x, u(x), v_i(x)) \leq A_\infty (1 + |v_i(x)|^q),
\]
where \( A_\infty := \sup \{a(x,u) : x \in \Omega, \ |u| \leq |u|_\infty \} < \infty \), and thus equi-integrability of
\( \{f(x,u,v_i)\} \) follows from the \( q \)-equi-integrability of \( \{v_i\} \) over \( D \). Moreover in the remaining
of the proof of Theorem 1.1, the growth condition (3.6) should be replaced by
\[
0 \leq g(x,v) \leq A_\infty (1 + |v|^p)
\]
(3.19) for a.e. \( x \in \Omega \) and all \( v \in \mathbb{R}^d \).

**Step 3:** If \( 1 \leq p \leq \infty \) and \( q = \infty \) then in Lemma 3.1 the hypothesis (3.1) should be replaced by
\[
\begin{align*}
    u_k &\to u \text{ in } L^p(D; \mathbb{R}^m), \\
    \hat{v}_k &\rightharpoonup v \text{ in } L^\infty(D; \mathbb{R}^d), \\
    A\hat{v}_n &\to 0 \text{ in } L^r(D; \mathbb{R}^d) \text{ for some } r > N,
\end{align*}
\]
the growth condition should be replaced by (3.19) if \( 1 \leq p < \infty \), \( q = \infty \), and by \( g \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R}^d; [0,\infty)) \) if \( p = q = \infty \), and we can proceed similarly to the proof of Lemma 3.1 to show that
\( w_{i,n} \rightharpoonup 0 \) in \( L^\infty(D; \mathbb{R}^d) \) and \( Au_{i,n} \to 0 \) in \( L^r(D; \mathbb{R}^d) \), and use Proposition 2.3 (iii) to get
\[
\|v_i - \hat{v}_n\|_{\infty} \to 0.
\]
We omit the details. \( \square \)

## 4 Proofs of Theorems 1.3 and 1.5

**Proof of Theorem 1.3.** We present the proof for \( 1 \leq p < \infty \), the case \( p = \infty \) being very similar.
Fix \( u \in W^{s,p}(\Omega; \mathbb{R}^n) \), and for \( D \in \mathcal{O}(\Omega) \) define
\[
\mathcal{F}(u; D) := \inf \left\{ \liminf_{k \to \infty} \int_D f(x, u_k, \ldots, \nabla^s u_k) \, dx : \{u_k\} \subset W^{s,p}(D; \mathbb{R}^n), \right. \\
\left. u_k \rightharpoonup u \text{ in } W^{s,p}(D; \mathbb{R}^n) \right\},
\]
and let \( g \) be the Carathéodory function
\[
g(x,v) := f(x, u(x), \ldots, \nabla^{s-1} u(x), v).
\]
Reasoning as in Lemma 3.4, it is easy to show that $\mathcal{F}(u; \cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^N[\Omega]$.  

For any function $v \in L^p(\Omega; E^n)$ set  

$$  \mathcal{G}(v; D) := \inf \left\{ \liminf_{k \to \infty} \int_D g(x, V_k(x)) \, dx : \{V_k\} \subset L^p(D; E^n) \cap \ker \mathcal{A} \text{ is } p\text{-equi-integrable,}  
\quad \text{and } V_k \rightharpoonup v \text{ in } L^p(D; E^n) \right\},  $$  

where the differential operator $\mathcal{A}$ is given by  

$$  \mathcal{A}u := \left( \frac{\partial}{\partial x_i} v_{i_1 \ldots i_{h+1} \ldots i_s} - \frac{\partial}{\partial x_j} v_{i_1 \ldots i_{h+2} \ldots i_s} \right)_{0 \leq h \leq s-1, 1 \leq i_j \leq i_s \leq N}. $$  

Here $h = 0$ and $h = s - 1$ correspond to the multi-indices $j i_2 \ldots i_s$ and $i_1 \ldots i_{s-1} j$. By Theorem 3.6 (and Corollary 3.2), and where the target space $\mathbb{R}^d$ is being replaced by the finite dimensional vector space $E^n$, for any $D \in \mathcal{O}(\Omega)$  

$$  \mathcal{G}(v; D) = \int_D \mathcal{Q}_A g(x, v(x)) \, dx,  $$  

where for a.e. $x \in \Omega$ and for all $v \in E^n$,  

$$  \mathcal{Q}_A g(x, v) := \inf \left\{ \int_Q g(x, v + w(y)) \, dy : w \in C_{1\text{-per}}^\infty(\mathbb{R}^N; E^n) \cap \ker \mathcal{A}, \int_Q w(y) \, dy = 0 \right\}. $$  

As shown in [22],  

$$  \left\{ w \in C_{1\text{-per}}^\infty(\mathbb{R}^N; E^n) : A w = 0, \quad \int_Q w \, dx = 0 \right\} = \{ \nabla^s \varphi : \varphi \in C_{1\text{-per}}^\infty(\mathbb{R}^N; \mathbb{R}^n) \}.  $$  

Hence  

$$  \mathcal{Q}_A g(x, v) = \inf \left\{ \int_Q g(x, v + \nabla^s \varphi(y)) \, dy : \varphi \in C_{1\text{-per}}^\infty(\mathbb{R}^N; \mathbb{R}^N) \right\}. $$  

In particular  

$$  \mathcal{G}(\nabla^s u; D) = \int_D \mathcal{Q}_f f(x, u, \ldots, \nabla^s u) \, dx.  $$  

Let $\{u_k\} \subset W^{s,p}(\Omega; \mathbb{R}^n)$ be any sequence such that $u_k \rightharpoonup u$ in $W^{s,p}(\Omega; \mathbb{R}^n)$. Extracting a subsequence, if necessary, we may assume that  

$$  u_k := (u_k, \ldots, \nabla^{s-1} u_k) \rightharpoonup u := (u, \ldots, \nabla^{s-1} u) \quad \text{in } L^p(D; E^n). $$  

Since $\nabla^s u_k \rightharpoonup \nabla^s u$ in $L^p(D; E^n)$ and $A \nabla^s u_k = 0$, by Lemma 3.1 there exists a $p\text{-equi-integrable}$ sequence $\{V_k\} \subset L^p(D; E^n) \cap \ker \mathcal{A}$ such that $V_k \rightharpoonup \nabla^s u$ in $L^p(D; E^n)$ and  

$$  \liminf_{k \to \infty} \int_D g(x, V_k(x)) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} f(x, u_k, \ldots, \nabla^s u_k) \, dx. $$  

Thus  

$$  \mathcal{G}(\nabla^s u; D) \leq \mathcal{F}(u; D).  $$  

To prove the converse inequality, fix $x_0 \in \Omega$ and $r > 0$, and consider any $p\text{-equi-integrable}$ sequence $\{V_k\} \subset L^p(B(x_0; r); E^n) \cap \ker \mathcal{A}$ such that $V_k \rightharpoonup \nabla^s u$ in $L^p(B(x_0; r); E^n)$. An induction
argument, similar to the one used in [22] to prove (4.1) above, shows that \( A V_k = 0 \) if and only if there exists \( \varphi_k \in W^{s,p}(B(x_0;r); \mathbb{R}^n) \) such that \( \nabla^s \varphi_k = V_k \). By Lemmas 1.1–1.3 in [24], for any \( \varphi \in W^{s,p}(B(x_0;r); \mathbb{R}^n) \) we may find a unique function \( P \in C^\infty(\mathbb{R}^N; \mathbb{R}^n) \) whose components are polynomials of degree \( s-1 \) such that

\[
\int_{B(x_0,r)} \nabla^l (\varphi - P) \, dx = 0 \quad 0 \leq l \leq s-1,
\]

and a constant \( C(n, N, s, p, r) > 0 \) such that the following Poincaré type inequality holds

\[
\|\varphi - P\|_{W^{s,p}(B(x_0;r); \mathbb{R}^n)} \leq C\|\nabla^s \varphi\|_{L^p(B(x_0;r); E^s_n)}.
\]

Let \( P_k \) and \( P \) be the functions associated to \( \varphi_k \) and \( u \), respectively, and satisfying (4.4), (4.5). Since \( \nabla^s \varphi_k \rightharpoonup \nabla^s u \) in \( L^p(B(x_0;r); E^s_n) \), we have that

\[
\varphi_k - P_k \rightharpoonup u - P \text{ in } W^{s,p}(B(x_0;r); \mathbb{R}^n),
\]

so

\[
u_k := \varphi_k - P_k + P \rightharpoonup u \text{ in } W^{s,p}(B(x_0;r); \mathbb{R}^n).
\]

Consider a subsequence of \( \{V_k\} \) (not relabelled) such that the two sequences

\[
\{(u_k, \ldots, \nabla^{s-1} u_k, V_k)\} \text{ and } \{(u, \ldots, \nabla^{s-1} u, V_k)\}
\]
generate the Young measure \( \{\delta_{(u(x), \ldots, \nabla^{s-1} u(x)}) \otimes \nu_x \}_{x \in B(x_0,r)} \) and

\[(u_k, \ldots, \nabla^{s-1} u_k) \rightharpoonup (u, \ldots, \nabla^{s-1} u) \]

pointwise and in \( L^p(B(x_0;r); E^{s-1}_n) \). Since \( \{V_k\} \) is \( p \)-equi-integrable and \( u_k \) converge to \( u \) strongly in \( W^{s-1,p}_0(\Omega; \mathbb{R}^N) \), it follows from Theorem 2.1 and the growth condition on \( f \) that

\[
\lim_{k \to \infty} \int_{B(x_0,r)} f(u_k, \ldots, \nabla^{s-1} u_k, V_k) \, dx = \lim_{k \to \infty} \int_{B(x_0,r)} g(x, V_k(x)) \, dx.
\]

Thus

\[
\mathcal{G}(\nabla^s u; B(x_0,r)) \geq \mathcal{F}(u; B(x_0,r)),
\]

which, together with (4.3), yields

\[
\mathcal{G}(\nabla^s u; B(x_0,r)) = \mathcal{F}(u; B(x_0,r)).
\]

Since \( \mathcal{F}(u\cdot) \) and \( \mathcal{G}(\nabla^s u\cdot) \) are both traces of a Radon measures absolutely continuous with respect to \( L^N[\Omega] \), by (4.2) and (4.6) we immediately obtain that

\[
\mathcal{F}(u; D) = \mathcal{G}(\nabla^s u; D) = \int_D Q f(x, u, \ldots, \nabla^s u) \, dx.
\]

\[\square\]

Proof of Theorem 1.5. We only proof Theorem 1.5 for \( 1 \leq p < \infty \), the case \( p = \infty \) being very similar.

For \( v \in \mathbb{R}^{N-1} \) let

\[
v = (v^{(1)}, \ldots, v^{(N)}), \quad \text{where } v^{(i)} \in \mathbb{R}^N, \quad i = 1, \ldots, N-1, \quad v^{(N)} \in \mathbb{R}^{N-1}.
\]

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Given a function $v \in L^p(\Omega; \mathbb{R}^{N^2-1})$ define the differential operator $\mathcal{A}$ as follows

$$\mathcal{A}v := \begin{pmatrix}\text{curl } v^{(1)} \\ \vdots \\ \text{curl } v^{(N-1)} \\ \text{curl } (v^{(N)}, -v_1^{(1)} - \ldots - v_{N-1}^{(N-1)})\end{pmatrix}.$$ 

A straightforward calculation shows that $\mathcal{A}$ satisfies the constant-rank property (1.1).

Given a Carathéodory function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N^2} \to [0, \infty)$, we define $\hat{f}(x, u, v)$, for $(x, u, v) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N^2-1}$, as

$$\hat{f}(x, u, v) = f \left( x, u, \begin{pmatrix} v^{(1)}, \ldots, v^{(N-1)} \\ -v_1^{(1)} - \ldots - v_{N-1}^{(N-1)} \end{pmatrix} \right).$$

Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, with $\text{div } u = 0$, and let $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^N)$ be such that $\text{div } u_n = 0$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^N)$. By Lemma 3.1 there exists a $p$-equi-integrable sequence $\{V_n\} \subset L^p(\Omega; \mathbb{R}^{N^2-1}) \cap \ker \mathcal{A}$ such that $V_n \rightharpoonup v$ in $L^p(D; \mathbb{R}^{N^2-1})$ and

$$\liminf_{n \to \infty} \int_{\Omega} \hat{f}(x, u_n, v_n) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} f(x, u_n, \nabla u_n) \, dx = \liminf_{n \to \infty} \int_{\Omega} f(x, u_n, V_n) \, dx,$$  

where

$$v_n := \begin{pmatrix} \nabla u_1^{(1)} \\ \vdots \\ \nabla u_n^{(N-1)} \end{pmatrix}, \quad v := \begin{pmatrix} \nabla u_1^{(1)} \\ \vdots \\ \nabla u_n^{(N-1)} \end{pmatrix}. \quad (4.8)$$

Define

$$g(v; D) := \inf \left\{ \liminf_{n \to \infty} \int_D \hat{g}(x, V_n(x)) \, dx : \{V_n\} \subset L^p(D; \mathbb{R}^{N^2-1}) \cap \ker \mathcal{A} \text{ is } p\text{-equi-integrable,} \right\},$$

where $\hat{g}$ is the Carathéodory function defined by $\hat{g}(x, v) := \hat{f}(x, u(x), v)$. By Theorem 3.6 (and Corollary 3.2)

$$g(v; \Omega) = \int_{\Omega} \mathcal{Q} \hat{g}(x, v) \, dx, \quad (4.9)$$

where

$$\mathcal{Q} \hat{g}(x, v(x)) := \inf \left\{ \int_Q \hat{f}(x, u(x), v(x) + w(y)) \, dy : w \in C^{\infty}_{1\text{-per}}(\mathbb{R}^N; \mathbb{R}^{N^2-1}) \cap \ker \mathcal{A}, \right\}$$

$$\int_Q w(y) \, dy = 0.$$

Now

$$w \in C^{\infty}_{1\text{-per}}(\mathbb{R}^N; \mathbb{R}^{N^2-1}) \cap \ker \mathcal{A} \quad \text{and} \quad \int_Q w(y) \, dy = 0$$

if and only if there exists $\varphi \in C^{\infty}_{1\text{-per}}(\mathbb{R}^N; \mathbb{R}^N)$ such that

$$w := \begin{pmatrix} \nabla \varphi^{(1)} \\ \vdots \\ \nabla \varphi^{(N-1)} \end{pmatrix}, \quad \begin{pmatrix} \partial \varphi^{(N)}_1 \\ \vdots \\ \partial \varphi^{(N)}_{N-1} \end{pmatrix}.$$
and \( \frac{\partial \varphi^{(n)}}{\partial x} = -\frac{\partial \varphi^{(1)}}{\partial x} - \cdots - \frac{\partial \varphi^{(N-1)}}{\partial x} \). Hence

\[
\mathcal{Q}_A \hat{g}(x, u(x)) = \inf \left\{ \int_Q f(x, u(x), \nabla u(x) + \nabla \varphi(y)) \, dy : \varphi \in C_{1-per}^{\infty}(\mathbb{R}^N; \mathbb{R}^N), \right.
\]

\[
\text{div } \varphi = 0 \left\} = \tilde{f}(x, u(x), \nabla u(x)).
\]

Thus, by (4.7), (4.9), and (4.10),

\[
\int_{\Omega} \tilde{f}(x, u(x), \nabla u(x)) \, dx = G(v; \Omega) \leq \liminf_{n \to \infty} \int_{\Omega} \tilde{f}(x, u_n, V_n) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) \, dx,
\]

and, in turn,

\[
\int_{\Omega} \tilde{f}(x, u(x), \nabla u(x)) \, dx \leq \inf \left\{ \liminf_{n \to \infty} \int_{\Omega} f(x, u_n(x), \nabla u_n(x)) \, dx : \{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^N), \right.
\]

\[
\text{div } u_n = 0, \quad u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) \left\}.
\]

To prove the converse inequality, fix \( \varepsilon > 0 \). By the definition of \( G(v; \Omega) \), there exists a \( p \)-equi-integrable sequence \( \{V_n\} \subset L^p(D; \mathbb{R}^{N^2-1}) \cap \ker \mathcal{A} \) such that \( V_n \rightharpoonup v \) in \( L^p(D; \mathbb{R}^{N^2-1}) \) and

\[
\int_{\Omega} \tilde{f}(x, u(x), \nabla u(x)) \, dx + \varepsilon > \liminf_{n \to \infty} \int_D \hat{g}(x, V_n(x)) \, dx
\]

\[
= \liminf_{n \to \infty} \int_D \tilde{f}(x, u(x), V_n(x)) \, dx,
\]

where we used for \( v \) the notation introduced in (4.8). Now \( AV_n = 0 \) if and only if there exists \( \varphi_n \in W^{1,p}(\Omega; \mathbb{R}^N) \) such that

\[
V_n = \left( \nabla \varphi_n^{[1]}, \ldots, \nabla \varphi_n^{[N-1]} \right), \quad \left( \begin{array}{c}
\frac{\partial \varphi_n^{[1]}}{\partial x_1} \\
\vdots \\
\frac{\partial \varphi_n^{[N-1]}}{\partial x_{N-1}}
\end{array} \right)
\]

and \( \frac{\partial \varphi^{(n)}}{\partial x} = -\frac{\partial \varphi^{(1)}}{\partial x} - \cdots - \frac{\partial \varphi^{(N-1)}}{\partial x} \). Since \( \nabla \varphi_n \rightharpoonup \nabla u \) in \( L^p(\Omega; \mathbb{R}^{N^2}) \), we have that

\[
\varphi_n = \frac{1}{|\Omega|} \int_{\Omega} \varphi_n(x) \, dx \rightharpoonup U \text{ in } W^{1,p}(\Omega; \mathbb{R}^N),
\]

where \( U = u + c \) for some constant \( c \in \mathbb{R}^N \). So

\[
u_n := \varphi_n - \frac{1}{|\Omega|} \int_{\Omega} \varphi_n(x) \, dx - c \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N),
\]

and \( \text{div } u_n = 0 \). Consider a subsequence \( \{V_{n_k}\} \) of \( \{V_n\} \) such that

\[
\lim_{k \to \infty} \int_{\Omega} \tilde{f}(x, u(x), V_{n_k}(x)) \, dx = \liminf_{n \to \infty} \int_{\Omega} \tilde{f}(x, u(x), V_n(x)) \, dx
\]

and \( \{(u_{n_k}, V_{n_k})\} \) and \( \{(u, V_{n_k})\} \) generates the Young measure \( \{\delta_{u(x)} \otimes \nu_{x}\}_{x \in \Omega} \). Since \( \{V_{n_k}\} \) is \( p \)-equi-integrable and \( u_{n_k} \) converge to \( u \) strongly in \( L^p(\Omega; \mathbb{R}^N) \), it follows from Theorem 2.1 and the growth condition on \( f \) that

\[
\lim_{k \to \infty} \int_{\Omega} \tilde{f}(x, u(x), V_{n_k}(x)) \, dx = \lim_{k \to \infty} \int_{\Omega} \tilde{f}(x, u_{n_k}(x), V_{n_k}(x)) \, dx.
\]
By (4.11)
\[
\int_{\Omega} \tilde{f}(x, u(x), \nabla u(x)) \, dx + \varepsilon > \lim_{k \to \infty} \int_{\Omega} f(x, u_{nk}(x), V_{nk}(x)) \, dx
\]
\[
= \lim_{k \to \infty} \int_{\Omega} f(x, u_{nk}(x), \nabla u_{nk}(x)) \, dx
\]
\[
\geq \inf \left\{ \liminf_{n \to \infty} \int_{\Omega} f(x, u_{n}(x), \nabla u_{n}(x)) \, dx : \{u_{n}\} \subset W^{1, p}(\Omega; \mathbb{R}^{N}), \right.
\]
\[
\left. \text{div} u_{n} = 0, \quad u_{n} \to u \text{ in } W^{1, p}(\Omega; \mathbb{R}^{N}) \right\}
\]
It now suffices to let $\varepsilon \to 0^{+}$. \hfill \Box

5 Homogenization

In this section we will limit our analysis to the case where $1 < q < \infty$.

**Lemma 5.1.** Let $f : \mathbb{R}^{N} \times \mathbb{R}^{d} \to [0, \infty)$ be a continuous function satisfying $(A_{1})-(A_{2})$. Let \( v \in L^q(D; \mathbb{R}^{d}) \cap \ker A \), where $D \in \mathcal{O}(\Omega)$, $\varepsilon_{k} \to 0^{+}$, and let $\{\tilde{v}_{k}\} \subset L^q(D_{1}; \mathbb{R}^{d})$ be a sequence of functions such that

\[
\tilde{v}_{k} \to v \text{ in } L^q(D_{1}; \mathbb{R}^{d}), \quad A\tilde{v}_{k} \to 0 \text{ in } W^{-1, q}(D_{1}; \mathbb{R}^{d}),
\]

for some $D_{1} \in \mathcal{O}(\Omega)$, with $D_{1} \subset D$. Then we can find a $q$-equi-integrable sequence $\{v_{k}\} \subset L^q(D_{1}; \mathbb{R}^{d}) \cap \ker A$ such that $\int_{D} v_{k} \, dx = \int_{D} v \, dx$,

\[
v_{k} \to v \text{ in } L^q(D_{1}; \mathbb{R}^{d}), \quad \|v_{k} - v_{k}\|_{L^q(D_{1})} \to 0 \text{ for all } 1 \leq s < q
\]

and

\[
\liminf_{k \to \infty} \int_{D_{1}} f(x/\varepsilon_{k}, v_{k}(x)) \, dx \leq \liminf_{k \to \infty} \int_{D_{1}} f(x/\varepsilon_{k}, \tilde{v}_{k}(x)) \, dx,
\]

\[
\limsup_{k \to \infty} \int_{D \setminus D_{1}} |v_{k}(x)|^{q} \, dx \leq \int_{D \setminus D_{1}} |v(x)|^{q} \, dx.
\]

Moreover, if $D = Q$, then $v_{k} = v + w_{k}$, with $w_{k} \in L_{1, \text{per}}^{q}(\mathbb{R}^{N}; \mathbb{R}^{d}) \cap \ker A$.

**Remark 5.2.** Lemma 5.1 implies, in particular, that for every $v \in L^q(\Omega; \mathbb{R}^{d}) \cap \ker A$

\[
\Gamma - \liminf_{n \to \infty} \mathcal{F}_{\varepsilon_{n}}(v; D) = \inf \left\{ \liminf_{n \to \infty} \mathcal{F}_{\varepsilon_{n}}(v_{n}; D) : v_{n} \in L^q(D; \mathbb{R}^{d}) \cap \ker A, \right. \]

\[
\left. v_{n} \to v \text{ in } L^q(D; \mathbb{R}^{d}), \quad \int_{D} v_{n} \, dx = \int_{D} v \, dx \right\},
\]

and if $D = Q$ then

\[
\Gamma - \liminf_{n \to \infty} \mathcal{F}_{\varepsilon_{n}}(v; Q) = \inf \left\{ \liminf_{n \to \infty} \mathcal{F}_{\varepsilon_{n}}(v + w_{n}; Q) : w_{n} \in L_{1, \text{per}}^{q}(\mathbb{R}^{N}; \mathbb{R}^{d}) \cap \ker A, \right. \]

\[
\left. w_{n} \to 0 \text{ in } L^q(Q; \mathbb{R}^{d}), \quad \int_{Q} w_{n} \, dx = 0 \right\}.
\]
Proof of Lemma 5.1. Let \( g(x) := x \) in \( Q \) and extend it periodically to \( \mathbb{R}^N \) with period 1. Set \( g_k(x) := g(x/\varepsilon_k) \). Since \( \{g_k\} \) is bounded in \( L^\infty \) and \( \hat{v}_k \rightarrow v \) in \( L^q(D_1; \mathbb{R}^d) \), by Theorem 2.1 there exists a subsequence \( \{\varepsilon_n\} \) of \( \{\varepsilon_k\} \) such that

\[
\{(g_n(x), \hat{v}_n(x))\} \quad \text{generates a Young measure} \quad \{\nu_x\}
\]

and

\[
\lim_{n \to \infty} \int_{D_1} f \left( x/\varepsilon_n, \hat{v}_n(x) \right) \, dx = \liminf_{k \to \infty} \int_{D_1} f \left( x/\varepsilon_k, \hat{v}_k(x) \right) \, dx.
\]

For \( i \in \mathbb{N} \) let

\[
F_i := \left\{ x \in D_1 : \text{dist}(x, \partial D_1) < \frac{1}{i} \right\}
\]

and consider cut-off functions \( \theta_i \) with compact support in \( D_1 \) and such that \( \theta_i \equiv 1 \) in \( D_1 \setminus F_i \). Set \( w_i := \theta_i(\hat{v}_n - v) \in L^q(Q; \mathbb{R}^d) \). Then we can proceed as in the proof of Lemma 3.1 to find a \( q \)-equi-integrable sequence \( \{v_i := v + w_i\} \), where \( \{w_i\} \) satisfies (3.3), (5.1) holds, and the two sequences

\[
\{(g_n(x), v_i(x))\} \quad \text{and} \quad \{(g_n(x), \hat{v}_n(x))\}
\]

generate the same Young measure \( \{\nu_x\} \). Hence by Theorem 2.1

\[
\lim_{i \to \infty} \int_{D_1} f \left( x/\varepsilon_n, v_i(x) \right) \, dx = \int_{D_1} \left( \int_{\mathbb{R}^N \times \mathbb{R}^d} f(X, V) \, d\nu_x(X, V) \right) \, dx
\]

\[
\leq \lim_{i \to \infty} \int_{D_1} f \left( x/\varepsilon_n, \hat{v}_n(x) \right) \, dx = \liminf_{k \to \infty} \int_{D_1} f \left( x/\varepsilon_k, \hat{v}_k(x) \right) \, dx,
\]

where we have used (A2), and the facts that \( \{v_i(x)\} \) is \( q \)-equi-integrable over \( D_1 \), and that \( f \) is a continuous function.

To prove the second inequality in (5.2), we remark that by (3.3) and the fact that \( \bar{w}_i = \theta_i(\hat{v}_n - v) \equiv 0 \) outside \( D_1 \), we have for all \( 1 \leq s < q \)

\[ ||v_i - v||_{L^s(D \setminus D_1)} = ||\bar{w}_i - w_i||_{L^s(D \setminus D_1)} \rightarrow 0. \]

Hence \( \{v_i(x)\} \) generates the Young measure \( \{\mu_x = \delta_{v(x)}\} \) on \( D \setminus D_1 \), and since \( \{v_i\} \) is \( q \)-equi-integrable we have that

\[
\limsup_{i \to \infty} \int_{D \setminus D_1} |v_i(x)|^q \, dx = \int_{D \setminus D_1} |Y|^q \, d\mu_x(Y) \, dx = \int_{D \setminus D_1} |v(x)|^q \, dx.
\]

To complete the proof it suffices to define \( v_k := v_{n_i} \) for each \( n_i \leq k < n_{i+1} \). Clearly

\[
\liminf_{k \to \infty} \int_{D_1} f \left( x/\varepsilon_k, v_k(x) \right) \, dx \leq \liminf_{i \to \infty} \int_{D_1} f \left( x/\varepsilon_n, v_i(x) \right) \, dx.
\]

\[ \square \]

Lemma 5.3. Let \( \varepsilon_n \to 0^+ \) and let \( \mathcal{R}(\Omega) \) be the family of all finite unions of open cubes contained in \( \Omega \) and with vertices in \( \mathbb{Q}^N \). Then there exists a subsequence \( \{\varepsilon_{n_k}\} \) of \( \{\varepsilon_n\} \) such that the \( \Gamma \)-limit

\[ \Gamma = \lim_{k \to \infty} \mathcal{F}_{\varepsilon_{n_k}}(v; R) \]

exists for all \( v \in L^q(R; \mathbb{R}^d) \cap \text{ker} A \) and for all \( R \in \mathcal{R}(\Omega) \).

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Proof of Lemma 5.3. Fix $R \in \mathcal{R}(\Omega)$. For simplicity set $\mathcal{F}_n := \mathcal{F}_{\epsilon_n}$ and let $B$ denote the closed unit ball of $L^q(R; \mathbb{R}^d)$. For each $l \in \mathbb{N}$ consider

$$lB := \{ v \in L^q(R; \mathbb{R}^d) : \|v\|_{L^q} \leq l \}.$$ 

Since $q > 1$ the dual of $L^q(R; \mathbb{R}^d)$ is separable, and hence the space $lB$ endowed with the weak topology is metrizable. Let $d_i$ be any metric which generates the $L^q$-weak topology. Consider $l = 1$ and apply Proposition 2.4 to the sequence of functionals $\{\mathcal{F}_{n_j}(\cdot; R)\}$ restricted to $(B \cap \ker A, d_i)$. Then we can find an increasing sequence of integers $\{n_j\}$ such that

$$\Gamma(d_1) = \lim_{j \to \infty} \mathcal{F}_{n_j}(v; R)$$

exists for all $v \in B \cap \ker A$. We now proceed recursively, so that given $l \in \mathbb{N}$ we apply Proposition 2.4 to the sequence of functionals $\{\mathcal{F}_{n_j}(\cdot; R)\}$ restricted to $(lB \cap \ker A, d_i)$ to obtain a subsequence $\{n_j\}$ of $\{n_j\}$ such that

$$\Gamma(d_l) = \lim_{j \to \infty} \mathcal{F}_{n_j}(v; R)$$

exists for all $v \in lB \cap \ker A$. Let $n_k := n^l_k$. Since $\{n_k\}$ is a subsequence of all $\{n_j\}$ we have that for each $l \in \mathbb{N}$

$$\Gamma(d_l) = \lim_{k \to \infty} \mathcal{F}_{n_k}(v; R)$$

exists for all $v \in lB \cap \ker A$.

We claim that the $\Gamma$-limit

$$\Gamma = \lim_{k \to \infty} \mathcal{F}_{n_k}(v; R) \quad \text{(5.3)}$$

exists for all $v \in L^q(R; \mathbb{R}^d) \cap \ker A$. Indeed assume by contradiction that this is not the case. Then there exists $v \in L^q(R; \mathbb{R}^d) \cap \ker A$ for which

$$\mathcal{F}^-(v; R) := \Gamma - \liminf_{k \to \infty} \mathcal{F}_{n_k}(v; R) < \mathcal{F}^+(v; R) := \Gamma - \limsup_{k \to \infty} \mathcal{F}_{n_k}(v; R).$$

Let $v_k \in L^q(R; \mathbb{R}^d) \cap \ker A$ be such that $v_k \to v$ in $L^q(R; \mathbb{R}^d)$ and

$$\lim_{k \to \infty} \mathcal{F}_{n_k}(v_k; R) = \mathcal{F}^-(v; R).$$

Since $v_k \to v$ in $L^q(R; \mathbb{R}^d)$, we may find an integer $l_0$ such that $v_k, v \in l_0 B \cap \ker A$ for all $k \in \mathbb{N}$. Consequently

$$d_{l_0}(v_k, v) \to 0 \quad \text{as } k \to \infty,$$

and thus

$$\Gamma(d_{l_0}) - \liminf_{k \to \infty} \mathcal{F}_{n_k}(v; R) \leq \liminf_{k \to \infty} \mathcal{F}_{n_k}(v_k; R) = \mathcal{F}^- (v; R)$$

$$< \mathcal{F}^+(v; R) \leq \Gamma(d_{l_0}) - \limsup_{k \to \infty} \mathcal{F}_{n_k}(v; R),$$

which contradicts the existence of the $\Gamma$-limit $\Gamma(d_{l_0}) - \lim_{k \to \infty} \mathcal{F}_{n_k}(v; R)$, and where we have used the fact that

$$\mathcal{F}^+(v; R) = \inf \left\{ \limsup_{k \to \infty} \mathcal{F}_{n_k}(z_k; R) : z_k \to v \text{ in } L^q(R; \mathbb{R}^d) \cap \ker A \right\} \leq \Gamma(d_{l_0}) - \limsup_{k \to \infty} \mathcal{F}_{n_k}(v; R)$$

$$= \inf \left\{ \limsup_{k \to \infty} \mathcal{F}_{n_k}(z_k; R) : z_k \to v \text{ in } L^q(R; \mathbb{R}^d) \right\}. $$

Hence (5.3) holds. To conclude the proof of the lemma it suffices to observe that since the family $\mathcal{R}(\Omega)$ is countable, with a diagonal process it is possible to extract a further subsequence for which (5.3) holds for all $R \in \mathcal{R}(\Omega)$. 

$\Box$
Remark 5.4. The previous proof asserts that for any given \( D \in \mathcal{O}(\Omega) \) and \( \varepsilon_n \to 0^+ \) there exists a subsequence \( \{\varepsilon_{n_k}\} \) (depending on the particular set \( D \)) of \( \{\varepsilon_n\} \) such that such that the \( \Gamma \)-limit
\[
\Gamma = \lim_{k \to \infty} \mathcal{F}_{\varepsilon_{n_k}}(v; D)
\]
eexists for all \( v \in L^q(D; \mathbb{R}^d) \cap \ker A \).

Lemma 5.5. Assume that conditions \((A_1)-(A_2)\) hold. Given \( \varepsilon_n \to 0^+ \), let \( \{\varepsilon_{n_k}\} \) be as in Lemma 5.3, and for any \( D \in \mathcal{O}(\Omega) \) set
\[
\mathcal{F}^{-}(\cdot; D) := \Gamma - \liminf_{k \to \infty} \mathcal{F}_{\varepsilon_{n_k}}(\cdot; D).
\]
Then \( \mathcal{F}^{-}(v; \cdot) \) is the trace of a Radon measure.

Proof of Lemma 5.5. We start by establishing inner regularity. Precisely, we claim that for any \( v \in L^q(D; \mathbb{R}^d) \cap \ker A \) and \( D \in \mathcal{O}(\Omega) \)
\[
\mathcal{F}^{-}(v; D) = \sup \{ \mathcal{F}^{-}(v; R) : R \in \mathcal{R}(\Omega), R \subset D \} = \lim_{R \uparrow D} \mathcal{F}^{-}(v; R),
\]
(5.4)
where the limit is taken over all finite unions of cubes \( R \in \mathcal{R}(\Omega) \) with \( R \subset D \). For fixed \( \eta > 0 \) there exists \( \delta > 0 \) such that
\[
\int_{D_0} C(1 + |v(x)|^q) \, dx < \eta
\]
(5.5)
for any measurable set \( D_0 \subset D \), with \( |D_0| < \delta \), and where \( C \) is the constant given in \((A_2)\).
Let \( R \in \mathcal{R}(\Omega) \), with \( R \subset D \) and \( |D \setminus R| < \delta \), and, in light of Lemma 5.3, consider a sequence
\[\{v_k\} \subset L^q(R; \mathbb{R}^d) \cap \ker A, \ \text{with} \ v_k \to v \ \text{in} \ L^q(R; \mathbb{R}^d), \text{and such that} \]
\[
\lim_{k \to \infty} \mathcal{F}_{\varepsilon_{n_k}}(v_k; R) = \mathcal{F}^{-}(v; R).
\]
By Lemma 5.1 there exists a \( q \)-equi-integrable sequence \( \{v_k\} \subset L^q(D; \mathbb{R}^d) \cap \ker A \) such that
\[
v_k \rightharpoonup v \ \text{in} \ L^q(D; \mathbb{R}^d), \quad \int_D v_k \, dx = \int_D v \, dx,
\]
and
\[
\liminf_{k \to \infty} \int_R f(x/\varepsilon_{n_k}, v_k(x)) \, dx \leq \lim_{k \to \infty} \int_R f(x/\varepsilon_{n_k}, \check{v}_k(x)) \, dx,
\]
\[
\limsup_{k \to \infty} \int_{D \setminus R} |v_k(x)|^q \, dx \leq \int_{D \setminus R} |v(x)|^q \, dx.
\]
Hence
\[
\mathcal{F}^{-}(v; D) \leq \liminf_{k \to \infty} \int_D f(x/\varepsilon_{n_k}, v_k(x)) \, dx
\]
\[
\leq \lim_{k \to \infty} \int_R f(x/\varepsilon_{n_k}, \check{v}_k(x)) \, dx + \limsup_{k \to \infty} \int_{D \setminus R} C(1 + |v_k(x)|^q) \, dx
\]
\[
\leq \mathcal{F}^{-}(v; R) + \int_{D \setminus R} C(1 + |v(x)|^q) \, dx \leq \mathcal{F}^{-}(v; R) + \eta,
\]
where we have used \((A_2)\) and (5.5). Consequently
\[
\mathcal{F}^{-}(v; D) \leq \sup \{ \mathcal{F}^{-}(v; R) : R \in \mathcal{R}(\Omega), R \subset D \} + \eta,
\]
and letting $\eta \to 0^+$ we obtain one inequality in (5.4). To show the opposite inequality, note that if \( \{v_k\} \subset L^q(D; \mathbb{R}^d) \cap \ker A \), with $v_k \rightharpoonup v$ in $L^q(D; \mathbb{R}^d)$, then the restriction of $v_k$ to $R$ belongs to $L^q(R; \mathbb{R}^d) \cap \ker A$, and $v_k \rightharpoonup v$ in $L^q(R; \mathbb{R}^d)$. Therefore

$$
\mathcal{F}^{-}(v; R) \leq \liminf_{k \to \infty} \int_R f(x/\varepsilon_{n_k}, v_k(x)) \, dx \leq \liminf_{k \to \infty} \int_D f(x/\varepsilon_{n_k}, v_k(x)) \, dx
$$

and by taking the infimum over all such sequences we get that

$$
\mathcal{F}^{-}(v; R) \leq \mathcal{F}^{-}(v; D), \tag{5.6}
$$

and in turn (5.4) holds.

In order to prove that $\mathcal{F}^{-}(v; \cdot)$ is the trace of a Radon measure, as it is usual it suffices to prove subadditivity for nested sets (see [3, 21]). Let $B \subset C \subset D$. By (5.4) for fixed $\eta > 0$ we find $R \in \mathcal{R}(\Omega)$ such that $R \subset D$ and

$$
\mathcal{F}^{-}(v; D) \leq \eta + \mathcal{F}^{-}(v; R).
$$

Construct $R_1$, $R_2 \in \mathcal{R}(\Omega)$ with

$$
R \subset R_1 \cup R_2, \quad R_1 \subset D \setminus \overline{B} \quad \text{and} \quad R_2 \subset C.
$$

By (5.6) we have

$$
\mathcal{F}^{-}(v; D) \leq \mathcal{F}^{-}(v; R) \leq \eta + \mathcal{F}^{-}(v; R_1 \cup R_2), \tag{5.7}
$$

By the definition of $\Gamma$-convergence and Lemma 5.1 there exist $v_k \in L^q(R_1; \mathbb{R}^d) \cap \ker A$ and $w_k \in L^q(R_2; \mathbb{R}^d) \cap \ker A$, with $v_k \rightharpoonup v$ in $L^q(R_1; \mathbb{R}^d)$ and $w_k \rightharpoonup v$ in $L^q(R_2; \mathbb{R}^d)$, such that

$$
\mathcal{F}^{-}(v; R_1) = \lim_{k \to \infty} \mathcal{F}_{\varepsilon_{n_k}}(v_k; R_1), \quad \mathcal{F}^{-}(v; R_2) = \lim_{k \to \infty} \mathcal{F}_{\varepsilon_{n_k}}(w_k; R_2), \tag{5.8}
$$

where \( \{\varepsilon_{n_k}\} \) is a subsequence of \( \{\varepsilon_{n_k}\} \) and \( \{v_k\}, \{w_k\} \) are $q$-equi-integrable over $R_1$ and $R_2$, respectively. Let $\theta_j$ be smooth cut-off functions which are equal to 1 on $B$ and 0 on $D \setminus C$, and such that $|\{0 < \theta_j < 1\}| \to 0$ as $j \to \infty$. Set

$$
\tilde{V}_{j,k} := (1 - \theta_j)v_k + \theta_j w_k.
$$

For $j$ fixed

$$
\mathcal{A} \tilde{V}_{j,k} = (1 - \theta_j)\mathcal{A}v_k + \theta_j \mathcal{A}w_k - \sum_{i=1}^N A^{(i)} v_k \frac{\partial \theta_j}{\partial x_i} + \sum_{i=1}^N A^{(i)} w_k \frac{\partial \theta_j}{\partial x_i} \to 0
$$

in $W^{-1,q}(R_1 \cup R_2; \mathbb{R}^d)$ strong, because $\|v_k - w_k\|_{W^{-1,q}(\overline{B}; \mathbb{R}^d)} \to 0$ as $k \to \infty$. Diagonalize to get $\tilde{V}_{j} := \tilde{V}_{j,k}$, such that $\tilde{V}_{j} \rightharpoonup v$ in $L^q(R_1 \cup R_2; \mathbb{R}^d)$, $\mathcal{A} \tilde{V}_{j} \to 0$ in $W^{-1,q}(R_1 \cup R_2; \mathbb{R}^d)$.

By Lemma 5.1 we can find $V_j \in L^q(R_1 \cup R_2; \mathbb{R}^d) \cap \ker A$ such that $V_j \rightharpoonup v$ in $L^q(R_1 \cup R_2; \mathbb{R}^d)$ and

$$
\liminf_{j \to \infty} \int_{R_1 \cup R_2} f \left( x/\varepsilon_{n_j}, V_j(x) \right) \, dx \leq \liminf_{j \to \infty} \int_{R_1 \cup R_2} f \left( x/\varepsilon_{n_j}, \tilde{V}_{j}(x) \right) \, dx.
$$
Consequently, by (5.7)

\[
\mathcal{F}^-(v; D) \leq \eta + \mathcal{F}^-(R_1 \cup R_2; D) \leq \eta + \liminf_{j \to \infty} \int_{R_1 \cup R_2} f \left( \frac{x}{\bar{\epsilon}_{n_k}}, V_j(x) \right) \, dx
\]

\[
\leq \eta + \liminf_{j \to \infty} \int_{R_1} f \left( \frac{x}{\bar{\epsilon}_{n_k}}, V_j(x) \right) \, dx
\]

\[
\leq \eta + \limsup_{j \to \infty} \int_{R_1} f \left( \frac{x}{\bar{\epsilon}_{n_k}}, v_{\epsilon_j}(x) \right) \, dx
\]

\[
+ \limsup_{j \to \infty} \int_{R_2} f \left( \frac{x}{\bar{\epsilon}_{n_k}}, w_{\epsilon_j}(x) \right) \, dx
\]

\[
+ \limsup_{j \to \infty} \int_{\{0 < \theta_j < 1\}} C(1 + |w_{\epsilon_j}(x)|^q + |v_{\epsilon_j}(x)|^q)
\]

\[
\leq \eta + \mathcal{F}^-(v; R_1) + \mathcal{F}^-(v; R_2) \leq \eta + \mathcal{F}^-(v; D \setminus \overline{B}) + \mathcal{F}^-(v; C),
\]

where we have used (5.6) and the fact that in (5.8) inferior limits are actually limits. It now suffices to let \( \eta \to 0^+ \).

\[\square\]

**Lemma 5.6.** Under conditions \((A_1)-(A_2)\), for \(\mathcal{L}^N\) a.e. \(x_0 \in \Omega\) we have

\[
d\mathcal{F}^-(v; \cdot)(x_0) = f_{\hom}(v(x_0)).
\]

**Proof of Lemma 5.6.** We divide the proof in three steps.

Given \(\epsilon_n \to 0^+\), let \(\{\epsilon_{n_k}\}\) be as in Lemma 5.3. In order to simplify the notations, in the proof of this lemma we will represent \(\{\epsilon_{n_k}\}\) simply by \(\{\epsilon\}\).

**Step 1:** We claim that

\[
\mathcal{F}^-(v(- \cdot, x_0); D + x_0) = \mathcal{F}^-(v; D).
\]

The proof is similar to the one of Lemma 3.9 in [11]. We present it here for the convenience of the reader. Let \(v_\epsilon \in L^q(D; \mathbb{R}^d) \cap \ker A\) be such that \(v_\epsilon \to v\) in \(L^q(D; \mathbb{R}^d)\) and

\[
\mathcal{F}^-(v; D) = \liminf_{\epsilon \to 0^+} \mathcal{F}_\epsilon(v_\epsilon; D).
\]

Consider the sequence \(z_\epsilon := [x_0/\epsilon] \in \mathbb{Z}^N\), so that \(x_\epsilon := z_\epsilon \epsilon\) converges to \(x_0\). Here \([z] := ([z_1], \ldots, [z_N])\), with \([z_i]\) denoting the integer part of \(z_i \in \mathbb{R}\). By the periodicity of \(f\),

\[
\mathcal{F}_\epsilon(v_\epsilon; D) = \int_D f \left( \frac{x + x_\epsilon}{\epsilon}, v_\epsilon(x) \right) \, dx = \int_{D + x_\epsilon} f \left( \frac{y}{\epsilon}, v_\epsilon(y - x_\epsilon) \right) \, dy.
\]

Let \(B \subset D\). For \(\epsilon\) sufficiently small we have that \(D + x_\epsilon \supset B + x_0\), and thus

\[
\mathcal{F}_\epsilon(v_\epsilon; D) \geq \int_{B + x_0} f \left( \frac{y}{\epsilon}, v_\epsilon(y - x_\epsilon) \right) \, dy.
\]

Since \(v_\epsilon(- x_\epsilon) \to v(- x_0)\) in \(L^q(B + x_0; \mathbb{R}^d)\), and \(v_\epsilon(- x_\epsilon) \in L^q(B + x_0; \mathbb{R}^d) \cap \ker A\), by (5.9), (5.10), we obtain

\[
\mathcal{F}^-(v; D) \geq \mathcal{F}^-(v(- \cdot, x_0); B + x_0).
\]

By letting \(R \cap D + x_0, R \in \mathcal{R}(\Omega)\), setting \(B := R - x_0\) above, we obtain by (5.4)

\[
\mathcal{F}^-(v; D) \geq \mathcal{F}^-(v(- \cdot, x_0); D + x_0).
\]

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The converse inequality follows in a similar way.

**Step 2:** Next, we show that

\[
\frac{d\mathcal{F}^{-}(v_i \cdot)}{dL^N}(x_0) \geq f_{\text{hom}}(v(x_0)) \quad \text{for } L^N \text{ a.e. } x_0 \in \Omega.
\]

Fix \( x_0 \in \Omega \) such that

\[
\lim_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0, r)} |v(x) - v(x_0)|^q \, dx = 0 \quad (5.11)
\]

and

\[
\frac{d\mathcal{F}^{-}(v_i \cdot)}{dL^N}(x_0) = \lim_{r \to 0^+} \liminf_{\varepsilon \to 0^+} \frac{1}{r^N} \int_{rQ} f \left( \frac{r}{\varepsilon}, v_{\varepsilon, r}(x) \right) \, dx < \infty,
\]

where we have used Step 1 and Lemma 5.5, and where we have chosen the radii \( r \to 0^+ \) such that \( \mathcal{F}^{-}(v(-x_0); \partial(rQ)) = 0 \). Here \( v_{\varepsilon, r} \in L^q(rQ; \mathbb{R}^d) \cap \ker A \) and \( v_{\varepsilon, r} \to v(-x_0) \) in \( L^q(rQ; \mathbb{R}^d) \) as \( \varepsilon \to 0^+ \). Then

\[
\frac{d\mathcal{F}^{-}(v_i \cdot)}{dL^N}(x_0) = \lim_{r \to 0^+} \liminf_{\varepsilon \to 0^+} \int_{rQ} f \left( \frac{r}{\varepsilon}, v(x_0) + w_{\varepsilon, r}(y) \right) \, dy
\]

where \( w_{\varepsilon, r}(y) := v_{\varepsilon, r}(ry) - v(x_0) \). As in the proof of Lemma 3.5, we have that \( w_{\varepsilon, r} \to 0 \) in \( L^q(Q; \mathbb{R}^d) \) if we first let \( \varepsilon \to 0 \) and then \( r \to 0^+ \). Diagonalize to get \( \tilde{w}_k \in L^q(Q; \mathbb{R}^d) \cap \ker A \) such that \( \tilde{w}_k \to 0 \) in \( L^q(Q; \mathbb{R}^d) \),

\[
\frac{d\mathcal{F}^{-}(v_i \cdot)}{dL^N}(x_0) = \lim_{k \to \infty} \int_{rQ} f(s_k y, v(x_0) + w_k(y)) \, dy,
\]

and where \( s_k := 1/\varepsilon_k \to \infty \). By Lemma 5.1, applied to the Carathéodory function \( h(x, v) := f(x, v(x_0) + v) \), there exists a \( q \)-equi-integrable sequence \( \{w_k\} \subset L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker A \) such that

\[
\tilde{w}_k \to 0 \quad \text{in } L^q(Q; \mathbb{R}^d), \quad \int_{Q} \tilde{w}_k \, dy = 0,
\]

and

\[
\liminf_{k \to \infty} \int_{Q} f(s_k y, v(x_0) + w_k(y)) \, dy \leq \lim_{k \to \infty} \int_{Q} f(s_k y, v(x_0) + \tilde{w}_k(y)) \, dy.
\]

Consequently

\[
\frac{d\mathcal{F}^{-}(v_i \cdot)}{dL^N}(x_0) \geq \liminf_{k \to \infty} \int_{Q} f(s_k y, v(x_0) + w_k(y)) \, dy
\]

\[
\geq \liminf_{i \to \infty} \liminf_{k \to \infty} \int_{Q} f(s_k y, v(x_0) + \theta_i(y)w_k(y)) \, dy,
\]

where \( 0 \leq \theta_i \leq 1 \) are smooth cut-off functions with compact support in \( Q \) such that \( \theta_i \equiv 1 \) in \( (1 - 1/i)Q \), and where we used the \( q \)-equi-integrability of \( \{w_k\} \) and (A2). Then \( \theta_i w_k \to 0 \) in \( L^q(Q; \mathbb{R}^d) \) as \( k \to \infty \) and \( i \to \infty \), in this order, and

\[
\lim_{i \to \infty} \lim_{k \to \infty} A(\theta_i w_k) = 0 \quad \text{in } W^{-1,q}(Q; \mathbb{R}^d).
\]

Diagonalize to get \( U_i := \theta_i w_k \) extended by zero outside \( Q \), such that \( U_i \to 0 \) in \( L^q(Q; \mathbb{R}^d) \), \( AU_i \to 0 \) in \( W^{-1,q}(Q; \mathbb{R}^d) \) as \( i \to \infty \), and

\[
\liminf_{k \to \infty} \liminf_{i \to \infty} \int_{Q} f(s_k y, v(x_0) + \theta_i(y)w_k(y)) \, dy = \lim_{i \to \infty} \int_{Q} f(s_ki, v(x_0) + U_i(y)) \, dy.
\]
Thus

\[
\frac{d F^{-}(v; \cdot)}{d\mathcal{L}^{N}}(x_{0}) \geq \liminf_{i \to \infty} \frac{1}{s_{k_{i}}^{N}} \int_{s_{k_{i}} Q} f \left( x, v(x_{0}) + U_{i} \left( \frac{x}{s_{k_{i}}} \right) \right) \, dx
\]

\[
\geq \liminf_{i \to \infty} \frac{1}{s_{k_{i}}^{N}} \int_{(\lfloor s_{k_{i}} \rfloor + 1) Q} f \left( x, v(x_{0}) + U_{i} \left( \frac{x}{s_{k_{i}}} \right) \right) \, dx
\]

\[
- \limsup_{i \to \infty} \frac{1}{s_{k_{i}}^{N}} \int_{(\lfloor s_{k_{i}} \rfloor + 1) Q \setminus s_{k_{i}} Q} f \left( x, v(x_{0}) + U_{i} \left( \frac{x}{s_{k_{i}}} \right) \right) \, dx,
\]

where \([s_{k_{i}}]\) denotes the integer part of \(s_{k_{i}}\). We claim that the last limit is zero. Indeed

\[
\frac{1}{s_{k_{i}}^{N}} \int_{(\lfloor s_{k_{i}} \rfloor + 1) Q \setminus s_{k_{i}} Q} f \left( x, v(x_{0}) + U_{i} \left( \frac{x}{s_{k_{i}}} \right) \right) \, dx = \int_{(\lfloor s_{k_{i}} \rfloor + 1) Q \setminus s_{k_{i}} Q} f(s_{k_{i}} y, v(x_{0}) + U_{i}(y)) \, dy.
\]

Since \((\lfloor s_{k_{i}} \rfloor + 1)/s_{k_{i}} \to 1\), we have that

\[
\left[ (\lfloor s_{k_{i}} \rfloor + 1) Q \setminus s_{k_{i}} Q \right] = \left( (\lfloor s_{k_{i}} \rfloor + 1)/s_{k_{i}} \right)^{N} - 1 \to 0,
\]

and thus the claim follows from the \(q\)-equi-integrability of \(\{U_{i}\}\) and \((A_{2})\). Hence, setting

\[
m_{i} := 1/s_{k_{i}}, \quad n_{i} := \lfloor s_{k_{i}} \rfloor + 1 \in \mathbb{N},
\]

we obtain

\[
\frac{d F^{-}(v; \cdot)}{d\mathcal{L}^{N}}(x_{0}) \geq \liminf_{i \to \infty} \frac{1}{n_{i}^{q}} \int_{n_{i} Q} f(x, v(x_{0}) + U_{i}(m_{i} x)) \, dx
\]

\[
= \liminf_{i \to \infty} \int_{Q} f(n_{i} y, v(x_{0}) + U_{i}(n_{i} m_{i} y)) \, dy.
\]

We claim that

\[
U_{i}(n_{i} m_{i} \cdot) \to 0 \quad \text{in} \quad L^{q}(Q; \mathbb{R}^{d}), \quad AU_{i}(n_{i} m_{i} \cdot) \to 0 \quad \text{in} \quad W^{-1,q}(Q; \mathbb{R}^{d})
\]

as \(i \to \infty\). Assuming that the claim holds, by Lemma 5.1 there exists a \(q\)-equi-integrable sequence \(\{V_{i}\} \subset L_{1,\text{per}}^{q}(\mathbb{R}^{N}; \mathbb{R}^{d}) \cap \ker \mathcal{A}\) such that

\[
V_{i} \rightharpoonup 0 \quad \text{in} \quad L^{q}(Q; \mathbb{R}^{d}), \quad \int_{Q} V_{i} \, dy = 0,
\]

and

\[
\frac{d F^{-}(v; \cdot)}{d\mathcal{L}^{N}}(x_{0}) \geq \liminf_{i \to \infty} \int_{Q} f \left( n_{i} y, v(x_{0}) + U_{i}(n_{i} m_{i} y) \right) \, dy
\]

\[
\geq \liminf_{i \to \infty} \int_{Q} f \left( n_{i} y, v(x_{0}) + V_{i}(y) \right) \, dy
\]

\[
= \liminf_{i \to \infty} \frac{1}{n_{i}^{q}} \int_{n_{i} Q} f \left( x, v(x_{0}) + V_{i} \left( \frac{x}{n_{i}^{q}} \right) \right) \, dx \geq f_{\text{hom}}(v(x_{0})),
\]

and where we have used the facts that

\[
V_{i} \left( \frac{1}{n_{i}^{q}} \right) \in L_{1,\text{per}}^{q}(\mathbb{R}^{N}; \mathbb{R}^{d}) \cap \ker \mathcal{A}, \quad \int_{n_{i} Q} V_{i} \left( \frac{1}{n_{i}^{q}} y \right) \, dy = 0.
\]

Thus it remains to show (5.12). If \(\varphi \in C_{c}^{\infty}(Q)\) then

\[
\int_{Q} U_{i}(n_{i} m_{i} y) \varphi(y) \, dy = \frac{1}{(n_{i} m_{i})^{q}} \int_{n_{i} m_{i} Q} U_{i}(x) \varphi \left( \frac{x}{n_{i} m_{i}} \right) \, dx
\]

\[
= \frac{1}{(n_{i} m_{i})^{q}} \left( \int_{Q} U_{i}(x) \varphi(x) \, dx + \int_{Q} U_{i}(x) \left( \varphi \left( \frac{x}{n_{i} m_{i}} \right) - \varphi(x) \right) \, dx \right),
\]

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where we have used the fact that $U_i(x) \equiv 0$ in $n_i m_i Q \setminus Q$. Since $U_i \to 0$ in $L^q(Q; \mathbb{R}^d)$ and $n_i m_i \to 1$ the first integral on the right hand side of the previous inequality tends to zero as $i \to \infty$. By Hölder’s inequality

$$
\left| \int_Q U_i(x) \left( \varphi \left( \frac{x}{n_i m_i} \right) - \varphi(x) \right) \, dx \right| \leq \left( \sup_i \left\| U_i \right\|_{L^q(Q)} \right) \left( \int_Q \left| \varphi \left( \frac{x}{n_i m_i} \right) - \varphi(x) \right|^{q'} \, dx \right)^{1/q'}.
$$

Since $\varphi$ is bounded we can apply Lebesgue Dominated Convergence Theorem to conclude that the right hand side approaches zero as $i \to \infty$. In a similar way we can show that

$$
\mathcal{A} U_i(n_i m_i \cdot) \to 0 \quad \text{in } W^{-1,q}(Q; \mathbb{R}^d) \text{ as } i \to \infty.
$$

We omit the details.

**Step 3:** To conclude the proof of the lemma it remains to show that

$$
\frac{d\mathcal{F}^-}{d\mathcal{L}^N}(v_{\cdot \cdot}^\perp)(x_0) \leq f_{\text{hom}}(v(x_0)) \quad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega.
$$

By Remark 1.8(i), for any fixed $\eta > 0$ we may find $k \in \mathbb{N}$, $w \in L^\infty_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $\int_{kQ} w \, dx = 0$ and

$$
\frac{1}{k^N} \int_{kQ} f(x, v(x_0) + w(x)) \, dx \leq f_{\text{hom}}(v(x_0)) + \eta.
$$

(5.13)

For any fixed $r > 0$ and for any $n \in \mathbb{N}$, let $v_{n,r} \colon (x) := w(xnk/r)$. Then $u_{n,r} \in L^\infty(rQ; \mathbb{R}^d) \cap \ker \mathcal{A}$, $u_{n,r} \to 0$ in $L^q(rQ; \mathbb{R}^d)$ as $n \to \infty$, and by Step 1

$$
\frac{d\mathcal{F}^-}{d\mathcal{L}^N}(v_{\cdot \cdot}^\perp)(x_0) = \lim_{r \to 0^+} \frac{\mathcal{F}^-(v_{\cdot \cdot}^\perp ; rQ)}{r^N}
$$

$$
= \lim_{r \to 0^+} \inf_{n \to \infty} \liminf_{n \to \infty} \frac{1}{r^N} \int_{rQ} f \left( \frac{x}{\varepsilon_{n,r}^r}, v(x_0 + x) + u_{n,r}(x) \right) \, dx
$$

$$
= \lim_{r \to 0^+} \inf_{n \to \infty} \liminf_{n \to \infty} \int_{Q} f(kry, v(x_0 + r y) + w(nk y)) \, dy,
$$

where $\varepsilon_{n,r} := r/nk$. Since $f(\cdot, v)$ is $Q$-periodic, there exists $\delta > 0$ such that if $|v - v(x_0)| < \delta$ then

$$
\sup_{s \in \mathbb{R}^N} |f(s, v + w(s)) - f(s, v(x_0) + w(s))| < \eta.
$$

Setting $E_{r,\delta} := \{y \in Q : |v(x_0 + ry) - v(x_0)| \geq \delta\}$, we deduce that

$$
\frac{d\mathcal{F}^-}{d\mathcal{L}^N}(v_{\cdot \cdot}^\perp)(x_0) \leq \eta + \lim \sup \limsup_{n \to \infty} \int_{Q} f(kry, v(x_0) + w(nk y)) \, dy
$$

$$
\quad + \lim \sup \limsup_{n \to \infty} \int_{E_{r,\delta}} C(1 + |v(x_0 + r y)|^q + |w(nk y)|^q) \, dy
$$

$$
= \eta + \frac{1}{k^N} \int_{kQ} f(y, v(x_0) + w(y)) \, dy \leq f_{\text{hom}}(v(x_0)) + 2\eta,
$$

where we have used (5.13), the $kQ$-periodicity of the function $h(y) := f(y, v(x_0) + w(y))$, the equi-integrability of $\{u_{n,r}^q\}$, and the fact that (5.11) entails

$$
\lim_{r \to 0^+} |\{y \in Q : |v(x_0 + r y) - v(x_0)| \geq \delta\}| = 0.
$$

It suffices to let $\eta \to 0^+$.
Proof of Theorem 1.7. We claim that for any $\varepsilon_n \to 0^+$

$$F_{\text{hom}}(\cdot; D) = \Gamma - \liminf_{n \to \infty} F_{\varepsilon_n}(\cdot; D).$$

By Lemmas 5.5 and 5.6 we always have

$$F_{\text{hom}}(\cdot; D) \geq \Gamma - \liminf_{k \to \infty} F_{\varepsilon_{n_k}} \geq \Gamma - \liminf_{n \to \infty} F_{\varepsilon_n}(\cdot; D).$$

Thus assume for contradiction that there exists $\varepsilon_n \to 0^+$ and $v \in L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$F_{\text{hom}}(v; D) > \Gamma - \liminf_{n \to \infty} F_{\varepsilon_n}(v; D).$$

Let $\{v_n\} \subset L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$ be such that $v_n \to v$ in $L^q(D; \mathbb{R}^d)$ and

$$F_{\text{hom}}(v; D) > \lim inf_{n \to \infty} F_{\varepsilon_n}(v_n; D),$$

and choose a subsequence $\{\varepsilon_{n_{k_j}}\}$ such that

$$F_{\text{hom}}(v; D) > \lim_{k \to \infty} F_{\varepsilon_{n_{k_j}}}(v_{n_{k_j}}; D).$$

Then, by the previous lemmas, we can extract a further subsequence $\{\varepsilon_{n_{k_j}}\}$ such that

$$F_{\text{hom}}(v; D) = \Gamma - \liminf_{j \to \infty} F_{\varepsilon_{n_{k_j}}}(v; D) \leq \lim_{j \to \infty} F_{\varepsilon_{n_{k_j}}}(v_{n_{k_j}}; D) \leq F_{\text{hom}}(v; D),$$

which is a contradiction and proves the claim.

Hence it remains to show that, when $(A_3)$ holds, for any $\varepsilon_n \to 0^+$ and $v \in L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$

$$\Gamma - \limsup_{n \to \infty} F_{\varepsilon_n}(v; D) \leq F_{\text{hom}}(v; D).$$

By taking $w_n \equiv v$ and using $(A_2)$ we get

$$\Gamma - \limsup_{n \to \infty} F_{\varepsilon_n}(v; D) \leq \limsup_{n \to \infty} F_{\varepsilon_n}(w_n; D) \leq C \int_D (|v(x)|^q + 1)dx.$$  

Hence for any sequence $\{v_n\} \subset L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $v_n \to v$ in $L^q(D; \mathbb{R}^d)$ and

$$\limsup_{n \to \infty} F_{\varepsilon_n}(v_n; D) \leq C \int_D (|v(x)|^q + 1)dx,$$

by $(A_3)$ we get that $\sup_n \|v_n\|_{L^q(D)} \leq L < \infty$, where the constant $L$ depends only on the

constants in $(A_2), (A_3)$, and on $\|v\|_{L^q(\Omega; \mathbb{R}^d)}$. Using the notation introduced in the proof of Lemma 5.3, we conclude that

$$\Gamma - \limsup_{n \to \infty} F_{\varepsilon_n}(v; D) = \Gamma(d_L) - \limsup_{n \to \infty} F_{\varepsilon_n}(v; D).$$  

(5.14)

By Remark 5.4 and by Lemma 5.6, for any subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ there exists a subsequence $\{\varepsilon_{n_{k_j}}\}$ (depending on $D$) such that

$$\Gamma - \lim_{j \to \infty} F_{\varepsilon_{n_{k_j}}}(v; D) = F_{\text{hom}}(v; D).$$

By (5.14) this implies that

$$\Gamma(d_L) - \lim_{j \to \infty} F_{\varepsilon_{n_{k_j}}}(v; D) = F_{\text{hom}}(v; D).$$

We can now apply the second part of Proposition 2.4 in the metric space $(L^B, d_L)$ to conclude that

$$\Gamma(d_L) - \lim_{n \to \infty} F_{\varepsilon_n}(v; D) = F_{\text{hom}}(v; D).$$

□

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Corollary 5.7. Under hypotheses (A_1)-(A_3) the function \( f_{\text{hom}} \) is \( A \)-quasiconvex and the following asymptotic formula holds

\[
f_{\text{hom}}(\xi) = \lim_{T \to +\infty} \frac{1}{T^N} \inf \left\{ \int_Q f(x, \xi + v(x)) \, dx : v \in L^q_T, (\mathbb{R}^N; \mathbb{R}^d) \cap \ker A, \int_Q v(x) \, dx = 0 \right\}.
\]  

(5.15)

Proof of Corollary 5.7. It may be shown easily, via a diagonalization procedure and in view of the coercivity condition (A_3), that \( F_{\text{hom}}(\cdot; Q) \) is \( L^q \)-sequentially weakly lower semicontinuous in \( \ker A \). In particular, this entails \( A \)-quasiconvexity for \( f_{\text{hom}} \). Indeed, fix \( v \in \mathbb{R}^d \) and \( w \in L^q_T, (\mathbb{R}^N; \mathbb{R}^d) \cap \ker A \), with \( \int_Q w(y) \, dy = 0 \), and define \( w_n(x) := w(nx) \). Then \( w_n \in L^q(Q; \mathbb{R}^d) \cap \ker A, w_n \to 0 \) in \( L^q(Q; \mathbb{R}^d) \), and so

\[
f_{\text{hom}}(v) = F_{\text{hom}}(v; Q) \leq \lim_{n \to \infty} \inf \int_Q f_{\text{hom}}(v + w_n(x)) \, dx
\]

(5.15) follows by setting \( T = 1/\varepsilon \) and changing variables in the last expression. \( \square \)

Finally, using Theorem 1.7, (A_3), and recalling Remark 5.2, we conclude that

\[
f_{\text{hom}}(v) = \min \left\{ F_{\text{hom}}(v + w; Q) : w \in L^q_T, (\mathbb{R}^N; \mathbb{R}^d) \cap \ker A, \int_Q w(x) \, dx = 0 \right\}
\]

(5.16) follows by setting \( T = 1/\varepsilon \) and changing variables in the last expression.

As a corollary of Theorem 1.7, we obtain the following result via the same choice of the underlying operator \( A \) as in the proof of Theorem 1.5.

Theorem 5.8 (Homogenization with constraint on the divergence). Assume that conditions (A_1) – (A_3) hold, with \( d = N^2 \), and let \( F_\varepsilon \) be defined by

\[
F_\varepsilon(u; D) := \int_D f \left( \frac{x}{\varepsilon}, \nabla u \right) \, dx
\]

on functions \( u \in W^{1,q}(\Omega; \mathbb{R}^N) \) such that div \( u = 0 \). Then the \( \Gamma \)-limit

\[
F(u; D) := \Gamma(L^q) - \lim_{\varepsilon \to 0} F_\varepsilon(u; D) = \int_D f_{\text{hom}}(\nabla u) \, dx
\]

exists on functions \( u \in W^{1,q}(\Omega; \mathbb{R}^N) \) such that div \( u = 0 \), where

\[
f_{\text{hom}}(v) = \inf_{k \in \mathbb{N}} \frac{1}{k^{N}} \inf_{kQ} \left\{ \int_{kQ} f(x, v + \nabla w(x)) \, dx : w \in W^{1,q}_{k\text{-per}}(\mathbb{R}^N; \mathbb{R}^N), \text{div } w = 0 \right\}
\]

(5.16) follows by setting \( T = 1/\varepsilon \) and changing variables in the last expression.

Appendix

We prove that in Remark 1.2(iii) in the Introduction

\[
\mathcal{F}(v; (a, b)) = \mathcal{F}(v_1, v_2; (a, b)) = \int_a^b (\psi^{**}(v_1) + v_2^2) \, dx.
\]

(5.16)
where \( \psi^*(v_1) \) is the convex envelope of
\[
\psi(v_1) := \min \{ (v_1 - 1)^2, (v_1 + 1)^2 \} = \begin{cases} (v_1 + 1)^2 & \text{if } v_1 \geq 0, \\ (v_1 - 1)^2 & \text{if } v_1 < 0. \end{cases}
\]

Indeed, if \( v^n_1 \to v_1 \) in \( L^2(a, b) \), \( v^n_2 \to v_2 \) in \( L^2(a, b) \) and \( (v^n_2)' \to 0 \) in \( H^{-1}(a, b) \) then the function \( v_2 \) is constant and Jensen's inequality yields
\[
\liminf_{n \to \infty} \int_a^b f(v^n_1, v^n_2) \, dx \geq \liminf_{n \to \infty} \int_a^b \left[ \psi(v^n_1) + (v^n_2)^2 \right] \, dx \\
\geq \liminf_{n \to \infty} \int_a^b \psi(v^n_1) \, dx + \liminf_{n \to \infty} \int_a^b (v^n_2)^2 \, dx \\
\geq \int_a^b \psi^*(v_1) \, dx + (v_2)^2(b - a).
\]

The arbitrariness of the sequence \( \{(v^n_1, v^n_2)\} \) allows us to conclude that
\[
\mathcal{F}(v_1; (a, b)) \geq \int_a^b \psi^*(v_1) \, dx + (v_2)^2(b - a).
\]

Conversely, suppose that if \( v_1 \) is smooth, \( v_1 \in L^2(a, b) \), and \( |\{x \in (a, b) : v_1(x) = 0\}| = 0 \), \( v_2 \in \mathbb{R} \), then
\[
\mathcal{F}(v_1; (a, b)) \leq \int_a^b \psi(v_1) \, dx + (v_2)^2(b - a), \quad (5.17)
\]

Then this inequality remains true for \( v_1 \in L^2(a, b) \), \( v_2 \in \mathbb{R} \) arbitrary, because we may approximate \( v_1 \) in \( L^2 \) strong by a sequence \( \{v^n_1\} \subset L^2(a, b) \cap C^\infty(a, b) \), \( |\{x \in (a, b) : v^n_1(x) = 0\}| = 0 \), and
\[
v_1 \to \mathcal{F}((v_1, v_2); (a, b)), \quad v_1 \to \int_a^b \psi(v_1) \, dx
\]

are, respectively, \( L^2 \)-weak lower semicontinuous and \( L^2 \)-strong continuous. Once we establish (5.17) for \( (v_1, v_2) \in L^2(a, b) \times \mathbb{R} \) then (5.16) follows because, once again, \( \mathcal{F}(\cdot, v_2); (a, b)) \) is \( L^2 \)-weak lower semicontinuous.

Fix now \( v_1 \in L^2(a, b) \cap C^\infty(a, b) \), with \( |\{x \in (a, b) : v_1(x) = 0\}| = 0 \), and let \( v_2 \in \mathbb{R} \), \( \delta > 0 \) be fixed. Set
\[
A^-_\delta := \{x \in (a, b) : (v_1(x) - 1)^2 < (v_1(x) + 1)^2 - \delta \}, \\
A^+_\delta := \{x \in (a, b) : (v_1(x) - 1)^2 > (v_1(x) + 1)^2 + \delta \}.
\]

If for all \( \delta > 0 A^-_\delta = \emptyset \) then \( (v_1(x) - 1)^2 \geq (v_1(x) + 1)^2 \) for every \( x \in (a, b) \) and we choose \( w_n \notin \mathbb{Q} \) with \( w_n \to v_2 \). Then
\[
\mathcal{F}((v_1, v_2); (a, b)) \leq \liminf_{n \to \infty} \int_a^b f(v, w_n) \, dx \\
= \liminf_{n \to \infty} \int_a^b [(v_1(x) + 1)^2 + (w_n)^2] \, dx \\
\leq \int_a^b \psi(v_1) \, dx + (v_2)^2(b - a).
\]

Similarly (5.17) holds if for all \( \delta > 0 A^-_\delta \neq \emptyset \neq A^+_\delta \). Choose a cut-off function \( \varphi_\delta \in C^\infty((a, b); [0, 1]) \) such that \( \varphi_\delta \equiv 1 \) in \( A^-_\delta \), \( \varphi_\delta \equiv 0 \) in \( A^+_\delta \), and let \( z_n \in \mathbb{Q} \), \( w_n \notin \mathbb{Q} \), be such that with \( w_n, z_n \to v_2 \). Define
\[
v^{n, \delta}_2(x) := \varphi_\delta(x) z_n + (1 - \varphi_\delta(x)) w_n.
\]
Since \((v_2^{n,\delta})' = \varphi'(x)(z_n - w_n)\), it is clear that
\[
\lim_{\delta \to 0^+} \lim_{n \to \infty} \|v_2^{n,\delta} - v_2\|_{L^2} = \lim_{\delta \to 0^+} \lim_{n \to \infty} \|(v_2^{n,\delta})'\|_{H^{-1}} = 0.
\]
We have
\[
\mathcal{F}(v_1, v_2); (a, b)) \leq \lim \inf \lim \inf_{\delta \to 0^+ \ n \to \infty} \int_a^b f(v_1, v_2^{n,\delta}) \, dx
\]
\[
= \lim \inf \lim \inf_{\delta \to 0^+ \ n \to \infty} \left\{ \int_{A_1^\delta} f(v_1, z_n) \, dx + \int_{A_2^\delta} f(v_1, w_n) \, dx + \int_{(a, b) \setminus (A_{A_1^\delta} \cup A_{A_2^\delta})} f(v_1, v_2^{n,\delta}) \, dx \right\}
\]
\[
\leq \lim \inf \lim \inf_{\delta \to 0^+ \ n \to \infty} \left\{ \int_{A_1^\delta} [\psi(v_1) + (z_n)^2] \, dx + \int_{A_2^\delta} [\psi(v_1) + (w_n)^2] \, dx + C \left| (a, b) \setminus (A_{A_1^\delta} \cup A_{A_2^\delta}) \right| \right\}
\]
\[
= \lim \inf \lim \inf_{\delta \to 0^+ \ n \to \infty} \left\{ \int_{A_1^\delta \cup A_2^\delta} [\psi(v_1) + (v_2)^2] \, dx + C \left| (a, b) \setminus (A_{A_1^\delta} \cup A_{A_2^\delta}) \right| \right\}
\]
\[
= \int_a^b \psi(v_1) \, dx + (v_2)^2(b - a),
\]
where we have used the fact that
\[
\left| (a, b) \setminus (A_{A_1^\delta} \cup A_{A_2^\delta}) \right| = \left| \{ x \in (a, b) : (v_1 - 1)^2 - (v_1 + 1)^2 < \delta \} \right|
\]
\[
= \left| \{ x \in (a, b) : |v_1(x)| < \delta/2 \} \right|
\]
\[
\to \left| \{ x \in (a, b) : v_1(x) = 0 \} \right| = 0
\]
as \(\delta \to 0^+\).

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