Characterizing graphs of maximum principal ratio

Michael Tait and Josh Tobin*

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Abstract

The principal ratio of a connected graph, denoted $\gamma(G)$, is the ratio of the maximum and minimum entries of its first eigenvector. Cioabă and Gregory conjectured that the graph on $n$ vertices maximizing $\gamma(G)$ is a kite graph: a complete graph with a pendant path. In this paper we prove their conjecture.

1 Introduction

Several measures of graph irregularity have been proposed to evaluate how far a graph is from being regular. In this paper we determine the extremal graphs with respect to one such irregularity measure, answering a conjecture of Cioabă and Gregory [5].

All graphs in this paper will be simple and undirected, and all eigenvalues are of the adjacency matrix of the graph. For a connected graph $G$, the eigenvector corresponding to its largest eigenvalue, the principal eigenvector, can be taken to have all positive entries. If $x$ is this eigenvector, let $x_{\min}$ and $x_{\max}$ be the smallest and largest eigenvector entries respectively. Then define the principal ratio, $\gamma(G)$ to be

$$\gamma(G) = \frac{x_{\max}}{x_{\min}}.$$ 

Note that $\gamma(G) \geq 1$ with equality exactly when $G$ is regular, and it therefore can be considered as a measure of graph irregularity.

Let $P_r \cdot K_s$ be the graph attained by identifying an end vertex of a path on $r$ vertices to any vertex of a complete graph on $s$ vertices. This has been called a kite graph or a lollipop graph. Cioabă and Gregory [5] conjectured that the connected graph on $n$ vertices maximizing $\gamma$ is a kite graph. Our main theorem proves this conjecture for $n$ large enough.

**Theorem 1.** For sufficiently large $n$, the connected graph $G$ on $n$ vertices with largest principal ratio is a kite graph.

We note that Brightwell and Winkler [4] showed that a kite graph maximizes the expected hitting time of a random walk. Other irregularity measures for graphs have been well-studied. Bell [3] studied the irregularity measure $\epsilon(G) := \lambda_1(G) - \bar{d}(G)$, the difference between the spectral radius and the average degree of $G$. He determined the extremal graph over all (not necessarily connected) graphs on $n$ vertices and $e$ edges. It is not known what the extremal connected graph is, and Aouchiche et al [2] conjectured that this extremal graph is a ‘pineapple’:

*Both authors were partially supported by NSF grant DMS 1362650

{mtait, rjtobin}@math.ucsd.edu
a complete graph with pendant vertices added to a single vertex. Bell also studied the \textit{variance} of a graph,

$$\text{var}(G) = \frac{1}{n} \sum_{v \in V(G)} |d_v - \bar{d}|^2.$$  

Albertson \cite{Albertson1} defined a measure of irregularity by

$$\sum_{uv \in E(G)} |d(u) - d(v)|$$

and the extremal graphs were characterized by Hansen and Mélot \cite{HansenM}.  

Nikiforov \cite{Nikiforov} proved several inequalities comparing $\text{var}(G)$, $\epsilon(G)$ and $s(G) := \sum_v |d(u) - \bar{d}|$.

Bell showed that $\epsilon(G)$ and $\text{var}(G)$ are incomparable in general \cite{Bell}. Finally, bounds on $\gamma(G)$ have been given in \cite{Brualdi, Hansen1, Hansen2, Hansen3, Hansen4}.

## 2 Preliminaries

Throughout this paper $G$ will be a connected simple graph on $n$ vertices. The eigenvectors and eigenvalues of $G$ are those of the adjacency matrix $A$ of $G$. The vector $v$ will be the eigenvector corresponding to the largest eigenvalue $\lambda_1$, and we take $v$ to be scaled so that its largest entry is 1. Let $x_1$ and $x_k$ be the vertices with smallest and largest eigenvector entries respectively, and if several such vertices exist then we pick any of them arbitrarily. Let $x_1, x_2, \ldots, x_k$ be a shortest path between $x_1$ and $x_k$. Let $\gamma(G)$ be the principal ratio of $G$. We will abuse notation so that for any vertex $x$, the symbol $x$ will refer also to $v(x)$, the value of the eigenvector entry of $x$. For example, with this notation the eigenvector equation becomes

$$\lambda v = \sum_{w \sim v} w.$$  

We will make use of the Rayleigh quotient characterization of the largest eigenvalue of a graph,

$$\lambda_1(G) = \max_{0 \neq v} \frac{v^T A(G) v}{v^T v}$$  \hspace{1cm} (1)$$

Recall that the vertices $v_1, v_2, \ldots, v_m$ are a \textit{pendant path} if the induced graph on these vertices is a path and furthermore if, in $G$, $v_1$ has degree 1 and the vertices $v_2, \ldots, v_{m-1}$ have degree 2 (note there is no requirement on the degree of $v_m$).

\textbf{Lemma 2.} If $\lambda_1 \geq 2$ and $\sigma = (\lambda_1 + \sqrt{\lambda_1^2 - 4})/2$, then for $1 \leq j \leq k$,

$$\gamma(G) \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1}.$$  

Moreover we have equality if the vertices $x_1, x_2, \ldots, x_j$ are a pendant path.

\textbf{Proof.} We have the following system of inequalities

\begin{align*}
\lambda_1 x_1 & \geq x_2 \\
\lambda_1 x_2 & \geq x_1 + x_3 \\
\lambda_1 x_3 & \geq x_2 + x_4 \\
& \vdots \\
\lambda_1 x_{j-1} & \geq x_{j-2} + x_{j-2}
\end{align*}
The first inequality implies that
\[ x_1 \geq \frac{1}{\lambda_1} x_2 \]
Plugging this into the second equation and rearranging gives
\[ x_2 \geq \frac{\lambda_1}{\lambda_1^2 - 1} x_3 \]
Now assume that
\[ x_i \geq \frac{u_{i-1}}{u_i} x_{i+1} \]
with \( u_j \) positive for all \( j < i \). Then
\[ \lambda_1 x_{i+1} \geq x_i + x_{i+2} \]
implies that
\[ x_{i+1} \geq \frac{u_i}{\lambda_1 u_i - u_{i-1}} x_{i+2} \]
where \( \lambda_1 u_i - u_{i-1} \) must be positive because \( x_j \) is positive for all \( j \). Therefore the coefficients \( u_i \) satisfy the recurrence
\[ u_{i+1} = \lambda_1 u_i - u_{i-1} \]
Solving this and using the initial conditions \( u_0 = 1, u_1 = \lambda \) we get
\[ u_i = \frac{\sigma^{i+1} - \sigma^{-i-1}}{\sigma - \sigma^{-1}} \]
In particular, \( u_i \) is always positive, a fact implicitly used above. Finally this gives,
\[ x_1 \geq \frac{u_0}{u_1} x_2 \geq \frac{u_0}{u_1} \cdot \frac{u_1}{u_2} x_3 \geq \cdots \geq \frac{x_j}{u_{j-1}} \]
Hence
\[ \gamma(G) = \frac{x_k}{x_1} = \frac{1}{x_1} \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1} \]
If these vertices are a pendant path, then we have equality throughout.

We will also use the following lemma which comes from the paper of Cioabă and Gregory.

**Lemma 3.** For \( r \geq 2 \) and \( s \geq 3 \),
\[ s - 1 + \frac{1}{s(s - 1)} < \lambda_1 (P_r \cdot K_s) < s - 1 + \frac{1}{(s - 1)^2}. \]

In the remainder of the paper we prove Theorem 1. We now give a sketch of the proof that is contained in Section 3.

1. We show that the vertices \( x_1, x_2, \ldots, x_{k-2} \) are a pendant path and that \( x_k \) is connected to all of the vertices in \( G \) that are not on this path (lemma 5).
2. Next we prove that the length of the path is approximately \( n - n/\log(n) \) (lemma 6).
3. We show that \( x_{k-2} \) has degree exactly 2 (lemma 9), which extends our pendant path to \( x_1, x_2, \ldots, x_{k-1} \). To do this, we find conditions under which adding or deleting edges increases the principal ratio (lemma 7).
4. Next we show that \( x_{k-1} \) also has degree exactly 2 (lemma 11). At this point we can deduce that our extremal graph is either a kite graph or a graph obtained from a kite graph by removing some edges from the clique. We show that adding in any missing edges will increase the principal ratio, and hence the extremal graph is exactly a kite graph.
3 Proof of Theorem

Let $G$ be the graph with maximal principal ratio among all connected graphs on $n$ vertices, and let $k$ be the number of vertices in a shortest path between the vertices with smallest and largest eigenvalue entries. As above, let $x_1, \ldots, x_k$ be the vertices of the shortest path, where $\gamma(G) = x_k/x_1$. Let $C$ be the set of vertices not on this shortest path, so $|C| = n - k$. Note that there is no graph with $n - k = 1$, as the endpoints of a path have the same principal eigenvector entry. Also $\lambda_1(G) \geq 2$, otherwise $P_{n-2} \cdot K_3$ would have larger principal ratio. Finally note that $k$ is strictly larger than 1, otherwise $x_k = x_1$ and $G$ would be regular.

Lemma 4. $\lambda_1(G) > n - k$.

Proof. Let $H$ be the graph $P_k \cdot K_{n-k+1}$. It is straightforward to see that in $H$, the smallest entry of the principal eigenvector is the vertex of degree 1 and the largest is the vertex of degree $n - k + 1$. Also note that in $H$, the vertices on the path $P_k$ form a pendant path. By maximality we know that $\gamma(G) \geq \gamma(H)$. Combining this with lemma 2 we get

$$\frac{\sigma^k - \sigma^-k}{\sigma - \sigma^-1} \geq \gamma(G) \geq \gamma(H) = \frac{\sigma^k_H - \sigma^-k_H}{\sigma_H - \sigma^-H}$$

where $\sigma_H = \left(\lambda_1(H) + \sqrt{\lambda_1(H)^2 - 4}\right)/2$.

Now the function

$$f(x) = \frac{x^k - x^-k}{x - x^-1}$$

is increasing when $x \geq 1$. Hence we have $\sigma \geq \sigma_H$, and so $\lambda_1(G) \geq \lambda_1(H) > n - k$. \hfill $\Box$

Lemma 5. $x_1, x_2, \ldots, x_{k-2}$ are a pendant path in $G$, and $x_k$ is connected to every vertex in $G$ that is not on this path.

Proof. By our choice of scaling, $x_k = 1$. From lemma 4

$$n - k < \lambda_1(G) = \sum_{y \sim x_k} y \leq |N(x_k)|.$$ 

Now $|N(x_k)|$ is an integer, so we have $|N(x_k)| \geq n - k + 1$. Moreover because $x_1, x_2, \ldots, x_k$ is an induced path, we must have that $|N(x_k)| = n - k + 1$ exactly, and hence the $N(x_k) = C \cup \{x_{k-1}\}$. It follows that $x_1, x_2, \ldots, x_{k-3}$ have no neighbors off the path, as otherwise there would be a shorter path between $x_1$ and $x_k$. \hfill $\Box$

Lemma 6. For the extremal graph $G$, we have $n - k = (1 + o(1)) \frac{n}{\log n}$.

Proof. Let $H$ be the graph $P_j \cdot K_{n-j+1}$ where $j = \left\lfloor n - \frac{n}{\log n} \right\rfloor$, and let $G$ be the connected graph on $n$ vertices with maximum principal ratio. Let $x_1, \ldots, x_k$ be a shortest path from $x_1$ to $x_k$ where $\gamma(G) = x_k/x_1$. By lemma 5, we have

$$\lambda_1(G) \leq \Delta(G) \leq n - k + 1.$$ 

By the eigenvector equation, this gives that

$$\gamma(G) \leq (n - k + 1)^k$$

Now, lemma 2 gives that

$$\gamma(H) = \frac{\sigma^j_H - \sigma^-j_H}{\sigma_H - \sigma^-H},$$
Lemma 7. Let \( \gamma \) be a graph obtained from \( G \) by adding some edges from \( x_{m-1} \) to \( V(G) \setminus \{x_1, \ldots, x_{m-1}\} \), where the addition of these edges does not affect which vertex has largest principal eigenvector entry. Let \( \lambda_1^+ \) be the largest eigenvalue of \( G_+ \) with leading eigenvector entry for vertex \( x \) denoted \( x^+ \), also normalized to have maximum entry one. Define \( \delta_1 \) and \( \delta_2 \) such that \( \lambda_1^+ = (1 + \delta_1) \lambda_1 \) and \( x^+_{m-1} = (1 + \delta_2) x_{m-1} \). Then

- \( \gamma(G_+) > \gamma(G) \) whenever \( \delta_1 > 4 \delta_2 / n \)
- \( \gamma(G_+) < \gamma(G) \) whenever \( \delta_1 \exp(2 \delta_1 \lambda_1 \log n) < \delta_2 / 3n \).

Proof. We have

\[
\sigma = \lambda_1 - \lambda_1^{-1} - \lambda_1^{-3} - 2 \lambda_1^{-5} - \cdots - \frac{2}{2n-3} \left( \frac{2n-2}{n} \right) \lambda_1^{-(2n-1)} - \cdots
\]

So

\[
\lambda_1^+ - \lambda_1 < \sigma_+ - \sigma < \lambda_1^+ - \lambda_1 - 2((\lambda_1^+)^{-1} - \lambda_1^{-1})
\]

when \( \lambda_1 \) is sufficiently large, which is guaranteed by lemma 6. Plugging in \( \lambda_1^+ = (1 + \delta_1) \lambda_1 \), we get

\[
\delta_1 \lambda_1 < \sigma_+ - \sigma < \delta_1 \lambda_1 + 2 \lambda_1^{-1} (1 - (1 + \delta_1)^{-1}) < \delta_1 \lambda_1 + \delta_1
\]

In particular

\[
(1 + \delta_1 / 2) \sigma < \sigma_+ < (1 + 2 \delta_1) \sigma
\]

To prove part (i), we wish to find a lower bound in the change in the first factor of equation (3). Let

\[
f(x) = \frac{x^{m-1} - x^{-m+1}}{x^{-1} - x^{-m}}.
\]
Then \(2mx^{m-3} > f'(x) > (m - 2)x^{m-3} - mx^{m-5}\), and using that \(n - m \sim n/\log(n)\) and \(\sigma \sim \lambda_1\) which goes to infinity with \(n\), we get \(f'(x) \gtrsim (m - 2)x^{m-3}\). By linearization and because \(f(\sigma) \sim \sigma^{m-2}\), it follows that

\[
\frac{\sigma_{+}^{-m-1} - \sigma_{+}^{-m+1}}{\sigma_{+} - \sigma_{+}^{-1}} \geq \left(1 + \frac{\delta_1(m-3)}{2}\right) \frac{\sigma^{m-1} - \sigma^{-m+1}}{\sigma - \sigma^{-1}}
\]

Hence, if

\[
\frac{\delta_1(m-3)}{2} > \delta_2
\]

then \(\gamma(G_+) > \gamma(G)\). In particular it is sufficient that \(\delta_1 > 4\delta_2/n\).

To prove part (ii), recall from above that \(f'(x) < 2mx^{m-3}\). Then, when \(x = (1 + o(1))(n/\log(n))\)

\[
f'(x + \varepsilon) < 2m(x + \varepsilon)^{m-3} = 2mx^{m-3}
\]

\[
(1 + \frac{\varepsilon}{x})^{m-3}
\]

\[
\leq 2mx^{m-3} \exp\left(\frac{m\varepsilon}{x}\right)
\]

\[
\leq 2nx^{m-3} \exp(2\log(n)\varepsilon)
\]

So for \(0 < \varepsilon < \delta_1\lambda_1\), we have

\[
f'(x + \varepsilon) < 2nx^{m-3} \exp(2\log(n)\delta_1\lambda_1)
\]

Hence

\[
(1 + 3n \exp(2\delta_1\lambda_1 \log n)\delta_1) \frac{\sigma_{+}^{m-1} - \sigma_{+}^{-m+1}}{\sigma - \sigma^{-1}} > \frac{\sigma_{+}^{m-1} - \sigma_{+}^{-m+1}}{\sigma_{+} - \sigma_{+}^{-1}}
\]

\[\square\]

**Lemma 8.** For every subset of \(U\) of \(N(x_k)\), we have

\[
|U| - 1 < \sum_{y \in U} y \leq |U|.
\]

An immediate consequence is that there is at most one vertex in the neighborhood of \(x_k\) with eigenvector entry smaller than \(1/2\).

**Proof.** The upper bound follows from \(y \leq 1\), and the lower bound from the inequalities

\[
\sum_{y \in N(x_k) \setminus U} y \leq |N(x_k)| - |U|
\]

and

\[
\sum_{y \in N(x_k)} y = \lambda_1(G) > |N(x_k)| - 1.
\]

\[\square\]

**Lemma 9.** The vertex \(x_{k-2}\) has degree exactly 2 in \(G\).
Proof. Assume to the contrary. Let $U = N(x_{k-2}) \cap N(x_k)$. Then $|U| \geq 2$, so by lemma 8 we have
\[ \sum_{y \in U} y > |U| - 1 \geq 1. \]
Now, by the same argument as in the proof of lemma 2, we have that
\[ \gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \left( \sum_{y \in U} y \right)^{-1} \]
Let $H = P_{k-1} \cdot K_{n-k+2}$. Then by maximality of $\gamma(G)$ we have
\[ \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} > \gamma(G) \geq \gamma(H) = \frac{\sigma_H^{k-1} - \sigma_H^{-k+1}}{\sigma_H - \sigma_H^{-1}} \]
So $\sigma > \sigma_H$, which means $\lambda_1(G) > \lambda_1(H) > n - k + 1$. This means that $\Delta(G) = n - k + 1$.

We now know that $x_1, x_2, \ldots, x_{k-1}$ is a pendant path in $G$, and so equation 3 becomes
\[ \gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \frac{1}{x_{k-1}} \tag{4} \]

Lemma 10. The vertex $x_{k-1}$ has degree less than $11|C|/\sqrt{\log n}$.

Proof. Assume to the contrary, so throughout this proof we assume that the degree of $x_{k-1}$ is at least $11|C|/\sqrt{\log n}$. Let $G_+$ the graph obtained from $G$ with an additional edge from $x_{k-1}$ to a vertex $z \in C$ with $z \geq 1/2$. Let $\lambda_1^+ = \lambda_1(G_+)$ and let $x^+$ be the principal eigenvector entry of vertex $x$ in $G_+$, where this eigenvector is normalized to have $x_k^+ = 1$.

Change in $\lambda_1$: By equation 1, we have $\lambda_1^+ - \lambda_1 \geq 2 \frac{x_k + z^+}{\|v\|^2}$. A crude upper bound on $\|v\|^2$ is
\[ \|v\|^2 \leq 1 + \sum_{y \sim x_k} y + \frac{2}{\lambda_1} + \frac{4}{\lambda_1^2} + \cdots < 2 \lambda_1 \]
We also have that $z \geq 1/2$ so
\[ \lambda_1^+ \geq \left( 1 + \frac{x_k - 1}{2 \lambda_1} \right) \lambda_1. \]

Change in $x_{k-1}$: Let $U = N(x_{k-1} \cap C)$. By the eigenvector equation we have
\[ x_{k-1} = \frac{1}{\lambda_1} \left( x_{k-2} + x_k + \sum_{y \in U} y \right) \]
\[ x_{k-1}^+ = \frac{1}{\lambda_1^+} \left( x_{k-2}^+ + x_k^+ + z^+ + \sum_{y \in U} y^+ \right) \]
Subtracting these, and using that $\lambda_1 < \lambda_1^+$ and $x_k = x_k^+ = 1$, we get
\[ x_{k-1}^+ - x_{k-1} \leq \frac{1}{\lambda_1} \left( x_{k-2}^+ - x_{k-2} + z^+ + \sum_{y \in U} y^+ - y \right). \]
By lemma 8, we have \( \sum_{y \in U} y^+ - y \leq 1 \). We also have \( x^+_k - x^+_{k-2} < 1 \) and \( x^+ \leq 1 \). Hence \( x^+_k - x^-_{k-1} \leq \frac{3}{\lambda_1} \), or
\[
  x^+_k \geq \left( 1 + \frac{3}{\lambda_1 x^-_{k-1}} \right) x^-_{k-1}
\]

We can only apply lemma 7 if \( x^+_k \) is the largest eigenvector entry in \( G_+ \). So we must consider two cases.

**Case 1:** If in \( G^+ \) the largest eigenvector entry is still attained by vertex \( x_k \), then we can apply lemma 7 and see that \( \gamma(G^+) > \gamma(G) \) if
\[
  \frac{x^-_{k-1}}{2\lambda_1^2} \geq \frac{12}{\lambda_1 x^-_{k-1} n}
\]
or equivalently
\[
  x^2_{k-1} \geq \frac{24\lambda_1}{n}.
\]

We have that \( \lambda_1 = (1 + o(1))(n - n/\log(n)) \), so it suffices for
\[
  x_{k-1} \geq \frac{5}{\sqrt{\log n}}.
\]

We know that
\[
  x_{k-1} > \frac{|U| - 1}{2\lambda_1}.
\]

By assumption
\[
  |U| + 2 = N(x_{k-1}) \geq 11|C|/\sqrt{\log n}
\]
Equation 5 follows from this, so \( \gamma(G^+) > \gamma(G) \).

**Case 2:** Say the largest eigenvector entry of \( G^+ \) is no longer attained by vertex \( x_k \). It is easy to see that the largest eigenvector entry is not attained by a vertex with degree less than or equal to 2, and comparing the neighborhood of any vertex in \( C \) with the neighborhood of \( x_k \) we can see that \( x_k \geq y \) for all \( y \in C \). So the largest eigenvector entry must be attained by \( x_{k-1} \). Then equation 4 no longer holds, instead we have
\[
  \gamma(G_+) = \frac{\sigma^+_{k-1} - \sigma^+_{k-1}}{\sigma^+_{k-1} - \sigma^+_{k-1}}.
\]

Recall that in lemma 7, we determined the change from \( \gamma(G_+) \) to \( \gamma(G) \) by considering \( \lambda_1^+ - \lambda_1 \) and \( x^+_k - x^-_{k-1} \). In this case, by (6), we must consider \( \lambda_1^+ - \lambda_1 \) and \( 1 - x_{k-1} \). Now if \( x^+_k > x^+_{k-1} \), then vertex \( x_{k-1} \) in \( G \) is connected to all of \( C \) except perhaps a single vertex. Hence in \( G \), the vertex \( x_{k-1} \) is connected to all of \( C \) except at most two vertices. This gives the bound
\[
  1 - x_{k-1} \leq \frac{3}{\lambda_1}
\]
and so as in the previous case, \( \gamma(G_+) > \gamma(G) \).

So in all cases, \( x_{k-1} \) is connected to all vertices in \( C \) that have eigenvector entry larger than 1/2. If all vertices in \( C \) have eigenvector entry larger than 1/2, then \( x_{k-1} \) is connected to all of \( C \), and this implies that \( x_{k-1} > x_k \), which is a contradiction. At most one vertex in \( C \) is smaller than 1/2, and so there is a single vertex \( z \in C \) with \( z < 1/2 \). We will quickly check that adding the edge \( \{x_{k-1}, z\} \) increases the principal ratio. As before let \( G_+ \) be the graph obtained by adding this edge. The largest eigenvector entry in \( G_+ \) is attained by \( x_{k-1} \), as its neighborhood strictly contains the neighborhood of \( x_k \). As above, adding the edge \( \{z, x_k\} \) increases the spectral radius at least
\[
  \lambda_1^+ > \left( 1 + \frac{z}{2\lambda_1} \right) \lambda_1
\]

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and we have $1 - x_{k-1} < 1 - z/\lambda_1$. Applying lemma 7, we see that $\gamma(G_+) > \gamma(G)$, which is a contradiction. Finally we conclude that the degree of $x_{k-1}$ must be smaller than $11|C|/\sqrt{\log n}$.

We note that this lemma gives that $x_{k-1} < 1/2$ which implies that any vertex in $C$ has eigenvector entry larger than $1/2$.

**Lemma 11.** The vertex $x_{k-1}$ has degree exactly 2 in $G$. It follows that $x_{k-1} < 2/\lambda_1$.

**Proof.** Let $U = N(x_{k-1}) \cap C$, $c = |U|$. If $c = 0$ then we are done. Otherwise let $G_-$ be the graph obtained from $G$ by deleting these $C$ edges. We will show that $\gamma(G_-) > \gamma(G)$.

(1) **Change in $\lambda_1$:** We have by equation 1,

$$\lambda_1 - \lambda_1^- \leq 2c x_{k-1} \|v\|_2^2$$

By Cauchy–Schwarz,

$$\|v\|_2^2 > \sum_{x \in N(x_k)} x^2 \geq \left(\frac{\sum_{x \in N(x_k)} x}{|C| + 1}\right)^2 \geq \frac{(n-k)^2}{n-k+1}$$

We also have

$$x_{k-1} \leq \frac{c + 2}{\lambda_1}$$

Combining these we get

$$\lambda_1 - \lambda_1^- < \frac{9c^2}{\lambda_1 (n-k+1)} \Rightarrow \lambda_1 < \left(1 + \frac{9c^2}{\lambda_1 \lambda_1^- (n-k+1)}\right) \lambda_1^-$$

We have $\lambda_1 \lambda_1^- > (n-k)^2$, so

$$\lambda_1 < \left(1 + \frac{10c^2}{(n-k)^3}\right) \lambda_1^-$$

(2) **Change in $x_{k-1}$:** At this point, we know that in $G_-$ the vertices $x_1, \ldots, x_k$ form a pendant path, and so by the proof of lemma 2, we have $x_{k-1}^- = (1 + o(1))/\lambda_1$. By the eigenvector equation and using that the vertices in $C$ have eigenvector entry at least $1/2$, we have $x_{k-1} > (1 + c/2)/\lambda_1$. So

$$x_{k-1} - x_{k-1}^- > \frac{1}{\lambda_1} \left(\frac{c}{2} + o(1)\right)$$

In particular,

$$x_{k-1} > \left(1 + \frac{c}{3x_{k-1}^- \lambda_1}\right) x_{k-1}^-$$

Applying lemma 7, it suffices now to show that

$$\frac{10c^2}{(n-k)^3} \exp\left(2 \frac{10c^2}{(n-k)^3} \lambda_1^- \log n\right) < \frac{c}{9x_{k-1}^- \lambda_1 n}.$$  \hspace{1cm} (7)

Now

$$\frac{10c^2}{(n-k)^3} < 10 \frac{11^2}{\log(n)} \frac{|C|^2}{(n-k)^3} < \frac{11^3}{\log n} \frac{\log n}{n} = \frac{11^3}{n}.$$
Similarly, \(2 \frac{10n^2}{(n-k)} \lambda_1 \log n < 2 \cdot 11^3\), so the left-hand side of equation 7 is smaller than \(C_0/n\), where \(C_0\) is an absolute constant. For the right-hand side, recall that \(x_{k-1}^k \lambda_1 = 1 + o(1)\), and also that
\[
c > \frac{11}{\sqrt{\log n}} \left( \frac{n}{\log n} + o(1) \right) > \frac{10n}{\log^{3/2} n}.
\]
So the right-hand side is larger than \(1/\log^{3/2} n\). Hence for large enough \(n\), the right-hand side is larger than the left-hand side.

We are now ready to prove the main theorem.

**Theorem 1.** For sufficiently large \(n\), the connected graph \(G\) on \(n\) vertices with largest principal ratio is a kite graph.

**Proof.** It remains to show that \(C\) induces a clique. Assume it does not, and let \(H\) be the graph \(P_k \cdot K_{n-k+1}\). We will show that \(\gamma(H) > \gamma(G)\), and this contradiction tells us that \(C\) is a clique. As before, lemma 2 gives that
\[
\gamma(H) = \frac{\sigma^k_H - \sigma^{-k}_H}{\sigma^1_H - \sigma^{-1}_H},
\]
where
\[
\sigma(H) = \lambda_1(H) - \sqrt{\lambda_1^2(H) - 4}.
\]
Since \(x_1, \ldots, x_k\) form a pendant path we also know that
\[
\gamma(G) = \frac{\sigma^k - \sigma^{-k}}{\sigma - \sigma^{-1}}.
\]

Now, \(\lambda_1(H) > \lambda_1(G)\) because \(E(G) \subsetneq E(H)\). Since the functions \(g(x) = x + \sqrt{x^2 - 4}\) and \(f(x) = (x^k - x^{-k})/(x - x^{-1})\) are increasing when \(x \geq 1\), we have \(\gamma(H) > \gamma(G)\).

\(\square\)

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**References**


